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JORDAN HIGHER DERIVATIONS REVISITED

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ABSTRACT. In this note, we consider the question of when a Jordan higher derivation of an arbitrary ring degenerates to a higher derivation. Furthermore, several applications to specific rings or algebras are presented.

1. Introduction

Let $\mathcal{A}$ be a ring (resp. algebra). An additive (resp. linear) mapping $d : \mathcal{A} \to \mathcal{A}$ is called a *derivation* if it satisfies the Leibniz rule $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{A}$. We say that an additive (resp. linear) mapping $d : \mathcal{A} \to \mathcal{A}$ is a *Jordan derivation* if it satisfies $d(x^2) = d(x)x + xd(x)$ for all $x \in \mathcal{A}$. When we study these operators, the standard problem is to find out whether a Jordan derivation is necessarily a derivation. We refer the readers to [2] and the references therein for more details.

Let $\mathbb{N}$ be the set of all non-negative integers and $D = \{d_n\}_{n\in\mathbb{N}}$ be a sequence of additive (resp. linear) mappings on $\mathcal{A}$ with $d_0 = \mathrm{id}_{\mathcal{A}}$ (the identity mapping of $\mathcal{A}$). If

$$d_n(xy) = \sum_{i=0}^{n} d_i(x)d_{n-i}(y)$$

for all $x, y \in \mathcal{A}$ and each non-negative integer n, then $D = \{d_n\}_{n\in\mathbb{N}}$ is called a *higher derivation* of $\mathcal{A}$. Let $\mathbb{K}$ be a field and $\mathcal{A} = \mathbb{K}[x_1, \cdots, x_n]$. The divided power differential operators $\frac{1}{n!}\frac{\partial^n}{\partial x_i^n}$ are basic examples of higher derivations of the polynomial algebras. This interesting notion of high derivations was introduced by Hasse and Schmidt in [6], and hence algebraists sometimes called them *Hasse-Schmidt derivations*. It should be remarked that the first component d_1 of each higher derivation is itself a derivation. Let $D = \{d_n\}_{n\in\mathbb{N}}$ be a sequence of additive (resp. linear) mappings on $\mathcal{A}$ with $d_0 = \mathrm{id}_{\mathcal{A}}$. Such sequence D is called a *Jordan higher derivations* of $\mathcal{A}$ if

$$d_n(x^2) = \sum_{i=0}^{n} d_i(x)d_{n-i}(x)$$

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for all $x \in \mathcal{A}$ and each non-negative integer n. Obviously, any higher derivation on $\mathcal{A}$ is a Jordan higher derivation. But the converse statements are not true in general. Therefore, it is natural to consider the question of when a Jordan higher derivation degenerates to a higher derivation.

The study of higher derivations are an active subject of research in both algebra and operator theory (see [5, 6, 7, 10, 11, 12, 14, 15, 16, 17, 22] and references therein). Note that higher derivations, as natural generalizations of derivations, are closely related to derivations. Heerema [8], Mirzavaziri [13] and Saymeh [18] independently proved that each higher derivation of a torsion free ring $\mathcal{A}$ is a combination of compositions of derivations, and hence one can characterize all higher derivations on $\mathcal{A}$ in terms of the derivations on $\mathcal{A}$. The second named author and Wei generalized this result to Jordan higher derivations [21]. Therefore, for a torsion free ring (especially for algebras over a field of characteristic zero) the study of Jordan higher derivations can turn into that of Jordan derivations.

On the other hand, we have known that there are several important rings (only need 2-torsion free) on which each Jordan higher derivation degenerates to a higher derivation. Such rings include full matrix rings over a commutative ring, (semi-)prime rings [4], triangular algebras [20] and so on. Hence the purpose of this paper is to find out some general criterions for an arbitrary ring such that every Jordan higher derivation on it is indeed a higher derivation. Although Jordan structures in associative rings have already been widely studied, there still are some interesting results which are needed to be revisited (see also [2, 3]).

This article is organized as following. In Section 2 we study Jordan higher derivations from the point of view of inner higher derivations. A connection between Jordan higher derivations and Jordan homomorphisms is shown in Section 3 as a direct generalization of one main result of Jacobson and Rickart [9]. Finally we consider the relationship between Jordan higher derivations and Hochschild 2-cocycles in Section 4.

2. From Inner Higher Derivations

For some technical convenience we always assume that *all algebras and rings in this paper are 2-torsion free*, i.e. $2x = 0$ implying that $x = 0$.

Let $\mathcal{A}$ be an associative algebra with identity element 1. Let $D = \{d_n\}_{n\in\mathbb{N}}$ be a sequence of linear mappings of $\mathcal{A}$. If there exist two sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}$ satisfying the conditions $a_0 = b_0 = 1$ and

$$\sum_{i=0}^{n} a_i b_{n-i} = \delta_{n0} = \sum_{i=0}^{n} b_i a_{n-i}$$

such that

$$d_n(x) = \sum_{i=0}^{n} a_i x b_{n-i} \tag{$\spadesuit$}$$

for all $x \in \mathcal{A}$ and each non-negative integer n, then it is easy to verify that $D = \{d_n\}_{n\in\mathbb{N}}$ is a higher derivation of $\mathcal{A}$, where δ_{n0} is the Kronecker sign. Higher derivations of the form ($\spadesuit$) are said to be *inner higher derivations*. It should be remarked that the main result of [19] shows the definition of inner higher derivation here is equivalent to that in the sense of Roy and Sridharan [17].

Nowicki proved in [14] that if every derivation of $\mathcal{A}$ is inner, then every higher derivation of $\mathcal{A}$ is inner (see also [19, Proposition 2.6]). Now we generalize this result to the case of Jordan derivations.

Lemma 2.1. *Let $D = \{d_n\}_{n\in\mathbb{N}}$ and $D' = \{d'_n\}_{n\in\mathbb{N}}$ be two Jordan higher derivations of $\mathcal{A}$. If $d_i = d'_i$ for all $i = 1, 2, \cdots, n-1$, then $d_n - d'_n$ is a Jordan derivation of $\mathcal{A}$.*

Proof. For any $x \in \mathcal{A}$, we have

$$\begin{aligned} d_n(x^2) - d'_n(x^2) &= d_n(x)x + xd_n(x) + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(x)d_j(x) \\ &\quad - d'_n(x)x - xd'_n(x) - \sum_{\substack{i+j=n \\ 0<i,j<n}} d'_i(x)d'_j(x) \\ &= d_n(x)x + xd_n(x) - d'_n(x)x - xd'_n(x) \\ &= (d_n - d'_n)(x)x + x(d_n - d'_n)(x). \end{aligned}$$

This shows $d_n - d'_n$ is a Jordan derivation of $\mathcal{A}$. □

Theorem 2.2. *If every Jordan derivation of $\mathcal{A}$ is inner, then every Jordan higher derivation of $\mathcal{A}$ is also inner.*

Proof. Let $D = \{d_n\}_{n\in\mathbb{N}}$ be a Jordan higher derivation of $\mathcal{A}$. Note that d_1 is a Jordan derivation of $\mathcal{A}$ and hence is an inner derivation by the hypothesis. We now assume that $\{d_i\}_{i=0}^k$ is an inner higher derivation of order k with $0 < k \leq n-1$. It follows that there exist two sequences $\{a_i\}_{i=0}^k$ and $\{b_i\}_{i=0}^k$ in $\mathcal{A}$ satisfying the conditions $a_0 = b_0 = 1$ and $\sum_{i+j=k} a_i b_j = \delta_{k0} = \sum_{i+j=k} b_i a_j$ such that

$$d_k(x) = \sum_{i+j=k} a_i x b_j$$

for all $x \in \mathcal{A}$ and each integer $0 < k \leq n-1$. For any element $a_n \in \mathcal{A}$, let us define

$$b_n := -a_n - \sum_{i=1}^{n-1} a_i b_{n-i}.$$

It is clear that $\sum_{i+j=n} a_i b_j = \delta_{n0} = 0$. Moreover, a direct calculation shows that $\sum_{i+j=n} b_i a_j = \delta_{n0} = 0$. Now we define

$$d'_n(x) := \sum_{i+j=n} a_i x b_j$$

for all $x \in \mathcal{A}$. Then $\{d_0 = \mathrm{id}_{\mathcal{A}}, d_1, \cdots, d_{n-1}, d'_n\}$ is an inner higher derivation of order n. It follows from Lemma 2.1 that $d_n - d'_n$ is a Jordan derivation. By the hypothesis, there exists an element $a'_n \in \mathcal{A}$ such that $d_n(x) = d'_n(x) + [a'_n, x]$ for all $x \in \mathcal{A}$. Therefore

$$\begin{aligned} d_n(x) &= d'_n(x) + a'_n x - x a'_n \\ &= (a_n + a'_n)x + x(b_n - a'_n) + \sum_{\substack{i+j=n \\ 0<i,j<n}} a_i x b_j \end{aligned}$$

for all $x \in \mathcal{A}$. We can show that $\{d_0 = \mathrm{id}_{\mathcal{A}}, d_1, \cdots, d_{n-1}, d_n\}$ is an inner higher derivation of order n and the inductive procedures complete the proof. □

Now we apply the main theorem of this section to some special algebras. Let $\mathcal{R}$ be a commutative ring with identity. We denote the set of all $n \times n$ upper triangular matrices over $\mathcal{R}$ by $T_n(\mathcal{R})$. The block upper triangular matrix algebra over $\mathcal{R}$ is denoted by $B_n^{\bar{d}}(\mathcal{R})$. We refer the reader to [1] for precise definition.

Corollary 2.3. *Every Jordan higher $\mathcal{R}$-derivation of $T_n(\mathcal{R})$ is inner.*

Proof. It is well-known that every Jordan derivation of $T_n(\mathcal{R})$ is a derivation (see [20, Corollary 2.6] for example) and hence is an inner derivation (see [19, Section 3]). Then the Theorem 2.2 makes every Jordan higher $\mathcal{R}$-derivation to be an inner higher derivation. □

Similarly, we have

Corollary 2.4. *Every Jordan higher $\mathcal{R}$-derivation of $B_n^{\bar{d}}(\mathcal{R})$ is inner.*

Corollary 2.5. *Every Jordan higher $\mathcal{R}$-derivation of the full matrix algebra $M_n(\mathcal{R})$ is inner.*

One may ask whether every Jordan higher derivation of $\mathcal{A}$ is a higher derivation provided that every Jordan derivation of $\mathcal{A}$ is a derivation? For the algebras over a field of characteristic zero, or the torsion free rings, this question has a positive answer [21]. However we do not know whether the statement holds or not for arbitrary rings.

3. From Jordan Homomorphisms

Let $\mathcal{A}$ and $\mathcal{B}$ be two rings with identity. Recall that an additive mapping $\varphi : \mathcal{A} \to \mathcal{B}$ is a Jordan homomorphism, if

$$\varphi(x^2) = \varphi(x)^2$$

for all $x \in \mathcal{A}$. The basic examples of Jordan homomorphisms are homomorphisms, anti-homomorphisms and their sum. Since all rings are 2-torsion free, there are

$$\varphi(xy + yx) = \varphi(x)\varphi(y) + \varphi(y)\varphi(x) \quad \text{and} \quad \varphi(xyx) = \varphi(x)\varphi(y)\varphi(x)$$

for all $x, y \in \mathcal{A}$. It was already observed by Jacobson and Rickart that the problem on Jordan derivations can be often reduced to the one on Jordan homomorphisms [9]. In fact, by a similar procedure [9, Theorem 22] can be extended to the case of Jordan higher derivations. We give out the proof here for completeness.

Theorem 3.1. *Let $\mathcal{A}$ be a ring with identity 1. If every Jordan homomorphism of $\mathcal{A}$ is the sum of a homomorphism and an anti-homomorphism, then every Jordan higher derivation of $\mathcal{A}$ is a higher derivation.*

Proof. Let $D = \{d_n\}_{n\in\mathbb{N}}$ be a Jordan higher derivation of $\mathcal{A}$. We use a recursive procedure to avoid some tedious computation. For any positive integer n, we can assume $d_k(xy) = \sum_{i+j=k} d_i(x)d_j(y)$ for all $x, y \in \mathcal{A}$ and any integer $1 \le k \le n-1$ by [9, Theorem 22].

For the positive integer n, let us define the mapping φ of $\mathcal{A}$ by

$$\varphi(a) = \begin{bmatrix} a & d_1(a) & d_2(a) & \dots & d_{n-1}(a) & d_n(a) \\ 0 & a & d_1(a) & \dots & d_{n-2}(a) & d_{n-1}(a) \\ \hdotsfor{6} \\ 0 & 0 & 0 & \dots & a & d_1(a) \\ 0 & 0 & 0 & \dots & 0 & a \end{bmatrix}$$

for all $a \in \mathcal{A}$. Then it is easy to check that φ is a Jordan homomorphism of $\mathcal{A}$. Denote the enveloping ring of $\varphi(\mathcal{A})$ by $\mathcal{U}$ and note that $\varphi(1)$ is the identity element of $\mathcal{U}$. By the hypothesis, φ is the sum of a homomorphism ϕ and an anti-homomorphism ψ. We claim that $\phi(1)$ and $\psi(1)$ are orthogonal central idempotents of $\mathcal{U}$. In fact, we only need to prove $\phi(1)$ and $\psi(1)$ are central. Note that homomorphisms and the negative of anti-homomorphisms are Lie homomorphisms. Therefore we have

$$\begin{aligned} [\phi(1), \varphi(a)] &= [\phi(1), \psi(a)] = [\varphi(1) - \psi(1), \psi(a)] \\ &= [-\psi(1), \psi(a)] = -[-\psi(1), -\psi(a)] \\ &= \psi([1, a]) = 0 \end{aligned}$$

for all $a \in \mathcal{A}$. Hence $\phi(1)$ and $\psi(1)$ are in the centre of $\mathcal{U}$. Furthermore

$$\begin{aligned} \varphi(a) &= \varphi(1)\varphi(a)\varphi(1) \\ &= \varphi(a)\phi(1) + \varphi(a)\psi(1) \end{aligned}$$

for all $a \in \mathcal{A}$. That means $\phi(a) = \varphi(a)\phi(1)$ is a homomorphism and $\psi(a) = \varphi(a)\psi(1)$ is an anti-homomorphism. Assume that

$$\phi(1) = \begin{bmatrix} e & a_{12} & a_{13} & \dots & a_{1,n+1} \\ 0 & e & a_{23} & \dots & a_{2,n+1} \\ 0 & 0 & e & \dots & a_{3,n+1} \\ \hdotsfor{5} \\ 0 & 0 & 0 & \dots & e \end{bmatrix}$$

and

$$\psi(1) = \begin{bmatrix} f & b_{12} & b_{13} & \dots & b_{1,n+1} \\ 0 & f & b_{23} & \dots & b_{2,n+1} \\ 0 & 0 & f & \dots & b_{3,n+1} \\ \hdotsfor{5} \\ 0 & 0 & 0 & \dots & f \end{bmatrix}.$$

It is clear that $1 = e + f$ and e, f are orthogonal central idempotents of $\mathcal{A}$. Using the fact that $\phi(a) = \varphi(a)\phi(1)$ is a homomorphism, we obtain

$$\begin{bmatrix} ab & d_1(ab) & d_2(ab) & \dots & d_{n-1}(ab) & d_n(ab) \\ 0 & ab & d_1(ab) & \dots & d_{n-2}(ab) & d_{n-1}(ab) \\ \hdotsfor{6} \\ 0 & 0 & 0 & \dots & ab & d_1(ab) \\ 0 & 0 & 0 & \dots & 0 & ab \end{bmatrix} \begin{bmatrix} e & a_{12} & a_{13} & \dots & a_{1,n+1} \\ 0 & e & a_{23} & \dots & a_{2,n+1} \\ 0 & 0 & e & \dots & a_{3,n+1} \\ \hdotsfor{5} \\ 0 & 0 & 0 & \dots & e \end{bmatrix} =$$

$$\begin{bmatrix} a & d_1(a) & \dots & d_n(a) \\ 0 & a & \dots & d_{n-1}(a) \\ \hdotsfor{4} \\ 0 & 0 & \dots & a \end{bmatrix} \begin{bmatrix} b & d_1(b) & \dots & d_n(b) \\ 0 & b & \dots & d_{n-1}(b) \\ \hdotsfor{4} \\ 0 & 0 & \dots & b \end{bmatrix} \begin{bmatrix} e & a_{12} & a_{13} & \dots & a_{1,n+1} \\ 0 & e & a_{23} & \dots & a_{2,n+1} \\ 0 & 0 & e & \dots & a_{3,n+1} \\ \hdotsfor{5} \\ 0 & 0 & 0 & \dots & e \end{bmatrix}$$

for all $a, b \in \mathcal{A}$. Considering the $(1, n+1)$-th entry of the above product on both side, we get

$$\sum_{i=0}^{n} d_i(ab) a_{i+1,n+1} = \sum_{i=0}^{n} \left(\sum_{k=0}^{i} d_k(a) d_{i-k}(b) \right) a_{i+1,n+1}$$

for all $a, b \in \mathcal{A}$, where we write $a_{n+1,n+1} = e$ for convenience. By the induction hypothesis, there is

$$d_n(ab)e = \sum_{i=0}^{n} d_i(a) d_{n-i}(b) e \tag{3.1}$$

for all $a, b \in \mathcal{A}$. Similarly, since $\psi(a) = \varphi(a)\psi(1)$ is an anti-homomorphism,

$$\begin{bmatrix} ab & d_1(ab) & d_2(ab) & \dots & d_{n-1}(ab) & d_n(ab) \\ 0 & ab & d_1(ab) & \dots & d_{n-2}(ab) & d_{n-1}(ab) \\ \hdotsfor{6} \\ 0 & 0 & 0 & \dots & ab & d_1(ab) \\ 0 & 0 & 0 & \dots & 0 & ab \end{bmatrix} \begin{bmatrix} f & b_{12} & b_{13} & \dots & b_{1,n+1} \\ 0 & f & b_{23} & \dots & b_{2,n+1} \\ 0 & 0 & f & \dots & b_{3,n+1} \\ \hdotsfor{5} \\ 0 & 0 & 0 & \dots & f \end{bmatrix} =$$

$$\begin{bmatrix} b & d_1(b) & \dots & d_n(b) \\ 0 & b & \dots & d_{n-1}(b) \\ \hdotsfor{4} \\ 0 & 0 & \dots & b \end{bmatrix} \begin{bmatrix} a & d_1(a) & \dots & d_n(a) \\ 0 & a & \dots & d_{n-1}(a) \\ \hdotsfor{4} \\ 0 & 0 & \dots & a \end{bmatrix} \begin{bmatrix} f & b_{12} & b_{13} & \dots & b_{1,n+1} \\ 0 & f & b_{23} & \dots & b_{2,n+1} \\ 0 & 0 & f & \dots & b_{3,n+1} \\ \hdotsfor{5} \\ 0 & 0 & 0 & \dots & f \end{bmatrix}$$

for all $a, b \in \mathcal{A}$. Therefore

$$\sum_{i=0}^{n} d_i(ab) b_{i+1,n+1} = \sum_{i=0}^{n} \left(\sum_{k=0}^{i} d_k(b) d_{i-k}(a) \right) b_{i+1,n+1} \tag{3.2}$$

for all $a, b \in \mathcal{A}$, where $b_{n+1,n+1} = f$. Note that $abf = baf$ holds true for any $a, b \in \mathcal{A}$ and f is a central idempotent. Multiplying the equation (3.2) by f and using the induction hypothesis, we obtain

$$d_n(ab)f = \sum_{i=0}^{n} d_i(a) d_{n-i}(b) f \tag{3.3}$$

for all $a, b \in \mathcal{A}$. Combining the identities (3.1) and (3.3) with the fact $1 = e + f$ yields that $d_n(ab) = \sum_{i=0}^{n} d_i(a)d_{n-i}(b)$. This completes the proof of the theorem. $\square$

It should be remarked that the Theorem 3.1 can be proved by a similar but much more complicated calculation instead of the induction. Let $\mathcal{A}$ be a ring with identity element and $M_m(\mathcal{A})$ be the full matrix ring over $\mathcal{A}$ with $m \geq 2$. For any derivation d of $\mathcal{A}$, we define $\overline{d} : M_m(\mathcal{A}) \to M_m(\mathcal{A})$ by $\overline{d}((a_{ij})) = (d(a_{ij}))$. Then $\overline{d}$ is a derivation of $M_m(\mathcal{A})$.

Corollary 3.2. *Every Jordan higher derivation $D = \{d_n\}_{n\in\mathbb{N}}$ of $M_m(\mathcal{A})$ is of the form*

$$d_n(x) = \sum_{i+j+k=n} A_i \overline{\delta}_j(x) B_k$$

for all $x \in M_m(\mathcal{A})$ and any non-negative integer n, where $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ are two sequences of matrices in $M_m(\mathcal{A})$ satisfying the conditions $A_0 = B_0 = I$ (the identity matrix) and

$$\sum_{i=0}^{n} A_i B_{n-i} = \delta_{n0} = \sum_{i=0}^{n} B_i A_{n-i}$$

and $\{\delta_n\}_{n\in\mathbb{N}}$ is a higher derivation of $\mathcal{A}$.

Proof. Since any Jordan homomorphism of $M_m(\mathcal{A})$ is the sum of a homomorphism and an anti-homomorphism by [9, Theorem 7], then Theorem 3.1 deduces that any Jordan higher derivation of $M_m(\mathcal{A})$ is in fact a higher derivation. Hence the desired from of $D = \{d_n\}_{n\in\mathbb{N}}$ follows from [11, Theorem 6.4]. $\square$

Note that Corollary 2.5 is a special case of Corollary 3.2.

4. From Hochschild 2-Cocycles

In this section, we will follow the steps of Brešar [2] to study Jordan higher derivations from the Hochschild 2-cocycles. Let $\mathcal{A}$ be a ring. Recall that a Hochschild 2-cocycle is a bi-additive mapping $(\ ,\ ) : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ satisfying

$$(xy, z) + (x, y)z = (x, yz) + x(y, z)$$

for all $x, y, z \in \mathcal{A}$. Let us consider $\mathcal{A}$ as a Lie ring under the bracket operation $[x, y] = xy - yx$. Set $\mathcal{A}^{(0)} = \mathcal{A}$ and define the series of derived Lie ring by $\mathcal{A}^{(n+1)} = [\mathcal{A}^{(n)}, \mathcal{A}^{(n)}]$ for every $n \geq 0$. Given a subset $\mathcal{S}$ of $\mathcal{A}$, we denote by $R(\mathcal{S})$ (resp. $I(\mathcal{S})$) the associative subring (resp. ideal) of $\mathcal{A}$ generated by $\mathcal{S}$. Note that $I(\mathcal{A}^{(n+1)}) \subseteq R(\mathcal{A}^{(n)})$ for every $n \geq 0$ (see the last paragraph of [2, P.412]).

Lemma 4.1. ([2, Theorem 2.1]) *If a skew-symmetric Hochschild 2-cocycle $(\ ,\ )$ of $\mathcal{A}$ satisfying*

$$(xy, x) + (x, y)x = 0$$

for all $x, y \in \mathcal{A}$, then $(R(\mathcal{A}^{(2)}), \mathcal{A}) = 0$.

Note that the ring $\mathcal{A}$ is 2-torsion free. The following lemma is well-known (see [4, Theorem 1.3 and Lemma 3.1]).

Lemma 4.2. *Let $D = \{d_n\}_{n\in\mathbb{N}}$ be a Jordan higher derivation of $\mathcal{A}$. Then for all $x, y, z \in \mathcal{A}$ and each $n \in \mathbb{N}$, we have*

(a) $d_n(xy + yx) = \sum_{i+j=n} \big(d_i(x)d_j(y) + d_i(y)d_j(x)\big)$;
(b) $d_n(xyx) = \sum_{i+j+k=n} d_i(x)d_j(y)d_k(x)$;
(c) $d_n(xyz + zyx) = \sum_{i+j+k=n} \big(d_i(x)d_j(y)d_k(z) + d_i(z)d_j(y)d_k(x)\big)$.

The main theorem of this section is

Theorem 4.3. *Let $\mathcal{A}$ be a ring such that $R(\mathcal{A}^{(2)}) = \mathcal{A}$ (in particular, if $I(\mathcal{A}^{(3)}) = \mathcal{A})$). Then Jordan higher derivation $D = \{d_n\}_{n\in\mathbb{N}}$ of $\mathcal{A}$ is a higher derivation.*

Proof. Note that d_1 is a Jordan derivation of $\mathcal{A}$ and hence d_1 is a derivation by [2, Corollary 3.2]. For any positive integer n, we assume that the relation $d_m(xy) = \sum_{i+j=m} d_i(x)d_j(y)$ holds true for all $x, y \in \mathcal{A}$ and all $m < n$. Define an bi-additive mapping $(\ ,\ )_n : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ by

$$(x, y)_n = d_n(xy) - \sum_{i+j=n} d_i(x)d_j(y)$$

for all $x, y \in \mathcal{A}$. It is clear that $(x, x)_n = 0$ and hence $(\ ,\ )_n$ is skew-symmetric. We claim that $(\ ,\ )_n$ is a Hochschild 2-cocycle. In light of the induction hypothesis, we have

$$\begin{aligned}
&(xy, z)_n + (x, y)_n z - (x, yz)_n - x(y, z)_n \\
=&d_n(xyz) - \sum_{i+j=n} d_i(xy)d_j(z) + d_n(xy)z - \sum_{i+j=n} d_i(x)d_j(y)z \\
&- d_n(xyz) + \sum_{i+j=n} d_i(x)d_j(yz) - xd_n(yz) + \sum_{i+j=n} xd_i(y)d_j(z) \\
=&-\sum_{\substack{i+j=n\\(i<n)}} d_i(xy)d_j(z) - \sum_{i+j=n} d_i(x)d_j(y)z \\
&+ \sum_{\substack{i+j=n\\(j<n)}} d_i(x)d_j(yz) + \sum_{i+j=n} xd_i(y)d_j(z) \\
=&-\sum_{\substack{i+j+l=n\\(0<l)}} d_i(x)d_j(y)d_l(z) - \sum_{i+j=n} d_i(x)d_j(y)z \\
&+ \sum_{\substack{i+k+l=n\\(0<i)}} d_i(x)d_k(y)d_l(z) + \sum_{i+j=n} xd_i(y)d_j(z) \\
=&-\sum_{\substack{i+j+l=n\\(0<i,l)}} d_i(x)d_j(y)d_l(z) - \sum_{\substack{j+l=n\\(0<l)}} xd_j(y)d_l(z) - \sum_{\substack{i+j=n\\(0<i)}} d_i(x)d_j(y)z - xd_n(y)z \\
&+ \sum_{\substack{i+k+l=n\\(0<i,l)}} d_i(x)d_k(y)d_l(z) + \sum_{\substack{i+k=n\\(0<i)}} d_i(x)d_k(y)z + \sum_{\substack{k+l=n\\(0<l)}} xd_k(y)d_l(z) + xd_n(y)z \\
=&0
\end{aligned}$$

for all $x, y, z \in \mathcal{A}$. On the other hand,

$$\begin{aligned}(xy, x)_n + (x, y)_n x &= d_n(xyx) - \sum_{i+j=n} d_i(xy)d_j(x) + d_n(xy)x - \sum_{i+j=n} d_i(x)d_j(y)x \\ &= \sum_{i+j+k=n} d_i(x)d_j(y)d_k(x) - \sum_{\substack{i+j=n \\ (0<j)}} d_i(xy)d_j(x) - d_n(xy)x \\ &\quad + d_n(xy)x - \sum_{i+j=n} d_i(x)d_j(y)x \\ &= \sum_{\substack{i+j+k=n \\ (0<k)}} d_i(x)d_j(y)d_k(x) + \sum_{i+j=n} d_i(x)d_j(y)x \\ &\quad - \sum_{\substack{i+j=n \\ (0<j)}} d_i(xy)d_j(x) - \sum_{i+j=n} d_i(x)d_j(y)x \\ &= 0\end{aligned}$$

for all $x, y \in \mathcal{A}$. Lemma 4.1 implies that $(R(\mathcal{A}^{(2)}), \mathcal{A})_n = 0$. In particular, $(I(\mathcal{A}^{(3)}), \mathcal{A})_n = 0$. Therefore $(\mathcal{A}, \mathcal{A})_n = 0$, i.e. $d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y)$ holds true for all $x, y \in \mathcal{A}$. □

We recall that a ring $\mathcal{A}$ is called a semiprime ring if there is no nonzero nilpotent ideal in $\mathcal{A}$. This definition is equivalent to $x\mathcal{A}x = 0$ implying $x = 0$. An ideal I of $\mathcal{A}$ is said to be *essential* if $I \cap J \neq 0$ for every nonzero ideal J of $\mathcal{A}$. In semiprime ring $\mathcal{A}$, I is an essential ideal if and only if I has trivial annihilator (see [2, P.415]). Now we can give a new proof of one old result of [4].

Corollary 4.4. *Let $\mathcal{A}$ be a 2-torsion free semiprime ring. Then every Jordan higher derivation of $\mathcal{A}$ is a higher derivation.*

Proof. Let $D = \{d_n\}_{n\in\mathbb{N}}$ be a Jordan higher derivation of $\mathcal{A}$. It is well-known that d_1 is a derivation (see [2, Corollary 3.8]). For any positive integer n, we assume that the relation $d_m(xy) = \sum_{i+j=m} d_i(x)d_j(y)$ holds true for all $x, y \in \mathcal{A}$ and all $m < n$. Define an bi-additive mapping $(\ ,\ )_n : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ by

$$(x, y)_n = d_n(xy) - \sum_{i+j=n} d_i(x)d_j(y)$$

for all $x, y \in \mathcal{A}$. It follows from the proof of Theorem 4.3 that $(R(\mathcal{A}^{(2)}), \mathcal{A})_n = 0$. In particular, $(I(\mathcal{A}^{(3)}), \mathcal{A})_n = 0$. Furthermore $I(\mathcal{A}^{(3)})(\mathcal{A}, \mathcal{A})_n = 0$ by [2, Remark 2.2]. Write $I = I(\mathcal{A}^{(3)})$ for convenience. Let $J = \{a \in \mathcal{A} | Ia = 0\}$ be the annihilator of I. Then it is clear that $\varepsilon = I \oplus J$ is an essential ideal since $\mathcal{A}$ is semiprime. Note that the annihilator of ε is zero. Hence we only need to prove $J(\mathcal{A}, \mathcal{A})_n = 0$.

For any $c \in Z(\mathcal{A})$, the centre of $\mathcal{A}$, it is clear that $d_1(c) \in Z(\mathcal{A})$ since every derivation is a Lie derivation. Assume $d_m(c) \in Z(\mathcal{A})$ for any integer $m < n$. We claim that $d_n(c) \in Z(\mathcal{A})$ for all $c \in Z(\mathcal{A})$. In fact, every Jordan higher derivation is a Lie triple higher derivation since $[[x, y], z] = x \circ (y \circ z) - y \circ (x \circ z)$ where

$x \circ y := xy + yx$. That means for any $c \in Z(\mathcal{A})$,

$$0 = d_n([[x,y],c]) = \sum_{i+j+k=n} [[d_i(x), d_j(y)], d_k(c)] = [[x,y], d_n(c)]$$

for all $x, y \in \mathcal{A}$. Hence $d_n(c) \in Z(\mathcal{A})$ by [2, Lemma 3.7].

Note that $J \subseteq Z(\mathcal{A})$ by [2, Lemma 3.7]. Picking $c \in J$ and $x \in \mathcal{A}$, we have

$$2d_n(cx) = d_n(cx + xc) = 2 \sum_{i+j=n} d_i(c) d_j(x).$$

Therefore $(J, \mathcal{A})_n = 0$ and hence $J(\mathcal{A}, \mathcal{A})_n = 0$ by [2, Remark 2.2]. □

Corollary 4.5. *Let $\mathcal{A}$ be a 2-torsion free semi-simple ring. Then every Jordan higher derivation of $\mathcal{A}$ is a higher derivation.*

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