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ON NEW FORMS OF THE RECIPROCITY THEOREMS

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Abstract. In his lost notebook, Ramanujan has stated a beautiful two variable reciprocity theorem. In the recent past, three and four variable generalizations of the reciprocity theorem of Ramanujan were established. In this paper we establish new symmetric and elegant forms of all the three reciprocity theorems. We also deduce therefrom some q-gamma, q-beta and eta function identities.

1. Introduction and preliminaries

On page 40 of his lost notebook [14], Ramanujan has given the following beautiful two variable reciprocity theorem.

$$\rho(a,b)-\rho(b,a)=\left(\frac{1}{b}-\frac{1}{a}\right)\frac{(aq/b,bq/a,q)_\infty}{(-aq,-bq)_\infty}, \tag{1.1}$$

where

$$\rho(a,b)=\left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-aq)_n} \tag{1.2}$$

and $a,\ b$ are complex numbers other than 0 and $-q^{-n}$.
Throughout this paper, we assume $|q|<1$ and employ the customary notations

$$(a)_\infty := (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n),$$

$$(a)_n := (a;q)_n := \frac{(a)_\infty}{(aq^n)_\infty},\ n \text{ is an integer},$$

$$(a_1,a_2,...,a_m)_n=(a_1)_n(a_2)_n...(a_m)_n,\ n \text{ is an integer or } \infty.$$

Andrews [4] was the first to establish (1.1) by employing his four-variable identity and the well-known Jacobi's triple product identity which is a special case of (1.1). Somashekara and Fathima [15] used Ramanujan's ${}_1\psi_1$ summation formula and Heine's transformation formula to establish an equivalent version of (1.1). Bhargava, Somashekara and Fathima [8] provided another proof of (1.1). Kim, Somashekara and Fathima [12] gave a proof of (1.1) using only q-binomial theorem. Guruprasad and Pradeep [10] also have devised a proof of (1.1) using q-binomial theorem. Adiga and Anitha [1] established a proof of (1.1) by the method of

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analytic continuation. Berndt, Chan, Yeap and Yee [7] found three different proofs of (1.1). Kang [11] constructed a proof of (1.1) along the lines of Venkatachaliengar's proof of Ramanujan's ${}_1\psi_1$ summation formula. In [17], Somashekara and Narasimha Murthy gave a proof of (1.1) using Abel's lemma and Jacobi's triple product identity. Recently, Somashekara, Narasimha Murthy and Shalini [19] have proved an equivalent form of the general identity of Andrews [4] using the parameter augumentation method and employed the same to derive (1.1). Further, Somashekara and Narasimha Murthy [18] have given a finite form of (1.1). For more details one may refer the book by Andrews and Berndt [5].

Kang [11] has established the following three and four variable generalizations of (1.1).

$$\rho_3(a,b;c)-\rho_3(b,a;c)=\left(\frac{1}{b}-\frac{1}{a}\right)\frac{(c,aq/b,bq/a,q)_\infty}{(-c/a,-c/b,-aq,-bq)_\infty}, \tag{1.3}$$

where

$$\rho_3(a,b;c):=\left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(c)_n(-1)^nq^{n(n+1)/2}a^nb^{-n}}{(-aq)_n(-c/b)_{n+1}} \tag{1.4}$$

and $|c|<|a|<1$ and $|c|<|b|<1$.

$$\begin{aligned}\rho_4(a,b;c,d)&-\rho_4(b,a;c,d)\\&=\left(\frac{1}{b}-\frac{1}{a}\right)\frac{(c,d,cd/ab,aq/b,bq/a,q)_\infty}{(-c/a,-c/b,-d/a,-d/b,-aq,-bq)_\infty},\end{aligned} \tag{1.5}$$

where

$$\begin{aligned}\rho_4&(a,b;c,d)\\&:=\left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(c,d,cd/ab)_n(1+cdq^{2n}/b)(-1)^nq^{n(n+1)/2}a^nb^{-n}}{(-aq)_n(-c/b,-d/b)_{n+1}}\end{aligned} \tag{1.6}$$

and $|c|,|d|<|a|,|b|<1$.

To derive (1.3), Kang [11] has employed Ramanujan's ${}_1\psi_1$ summation formula and Jackson's transformation of ${}_2\phi_1$ and ${}_2\phi_2$ series. Later, Adiga and Guruprasad [2] have given a proof of (1.3) using q-binomial theorem and q-Gauss summation formula. Somashekara and Mamta [16] have obtained (1.3) using (1.1) by parameter augmentation method. One more proof of (1.3) was given by Zhang [21]. Recently, in [19], Somashekara, Narasimha Murthy and Shalini have given a proof of (1.3) and in [18], Somashekara and Narasimha Murthy have given a finite form of (1.3).

Kang [11] has established the four variable reciprocity theorem (1.5) by employing Andrews generalization of ${}_1\psi_1$ summation formula [4, Theorem 6], Sear's transformation for ${}_3\phi_2$ series and a limiting case of Watson's transformation for a terminating very well-poised ${}_8\phi_7$ series. Adiga and Guruprasad [3] have derived (1.5) using an identity of Andrews [4, Theorem 1], Ramanujan's ${}_1\psi_1$ summation formula and Watson's transformation. Recently, Somashekara, Narasimha Murthy and Shalini [19] have given a proof of (1.5) and in [18], Somashekara and Narasimha Murthy have given a finite form of (1.5).

The main objective of this paper is to give new symmetric forms of the reciprocity theorems along with a unified approach to their proofs and to investigate their applications.

The paper is organised as follows. In section 2, we give some basic definitions and notations and state some results which we use in the remainder of the paper. In section 3, we state and prove the symmetric forms of the reciprocity theorems and in section 4, we present a number of applications of the main results.

2. Some standard definitions and notations

We recall some basic definitions and notations.
q-Gamma function [20] is defined by

$$\Gamma_q(x) = \frac{(q)_\infty}{(q^x)_\infty}(1-q)^{1-x},\ 0<q<1. \tag{2.1}$$

q-beta function [6] is defined by

$$B_q(x,y) = (1-q)\sum_{n=0}^{\infty}\frac{(q^{n+1})_\infty}{(q^{n+y})_\infty}q^{nx}. \tag{2.2}$$

We also have [6]

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \tag{2.3}$$

The Dedekind eta function is defined by

$$\eta(\tau) := e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi i n\tau}),\ Im(\tau)>0$$

$$:= q^{1/24}(q;q)_\infty,\ where\ e^{2\pi i\tau}=q. \tag{2.4}$$

A q-Shifted factorial identity [9, equation(I.17), p.352] is

$$(a)_{n+m} = (a)_m(aq^m)_n. \tag{2.5}$$

q-binomial theorem [9, equation(II.3), p.354] is given by

$$\sum_{n=0}^{\infty}\frac{(a)_n}{(q)_n}z^n = \frac{(az)_\infty}{(z)_\infty}. \tag{2.6}$$

Heine's transformation of ${}_2\phi_1$-series [9, equation(III.2), p.359] is

$$\sum_{n=0}^{\infty}\frac{(A,B)_n}{(q,C)_n}Z^n = \frac{(C/B,BZ)_\infty}{(C,Z)_\infty}\sum_{n=0}^{\infty}\frac{(ABZ/C,B)_n}{(q,BZ)_n}\left(\frac{C}{B}\right)^n. \tag{2.7}$$

Sear's transformation of ${}_3\phi_2$ series [9, equation(III.9), p.359] is

$$\sum_{n=0}^{\infty}\frac{(A,B,C)_n}{(q,D,E)_n}\left(\frac{DE}{ABC}\right)^n = \frac{(E/A,DE/BC)_\infty}{(E,DE/ABC)_\infty}\sum_{n=0}^{\infty}\frac{(A,D/B,D/C)_n}{(q,D,DE/BC)_n}\left(\frac{E}{A}\right)^n. \tag{2.8}$$

In [19], Somashekara, Narasimha Murthy and Shalini have established the following identity which is equivalent to the general identity of Andrews [4] [5, Theorem(6.2.1), p.114].

$$\sum_{n=0}^{\infty}\frac{(c,d)_n}{(-aq,-bq)_n}q^n = -\frac{(c,d)_\infty}{a(-aq,-bq)_\infty}\sum_{n=0}^{\infty}\frac{(-bq/d)_n}{(-c/a)_{n+1}}\left(\frac{-d}{a}\right)^n + \left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty}\frac{(c,-bq/d)_n}{(-bq)_n(-c/a)_{n+1}}\left(\frac{-d}{a}\right)^n. \tag{2.9}$$

3. Main results

In this section, we establish new symmetric forms of all the three reciprocity theorems.

Theorem 3.1. *If* $|a|,|b|<1$ *then*

$$f_2(a,b)-f_2(b,a)=\left(\frac{1}{a}-\frac{1}{b}\right)\frac{(q,aq/b,bq/a)_\infty}{(-a,-b)_\infty}, \tag{3.1}$$

where

$$f_2(a,b)=\frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{-q}{a}\right)_n(-b)^n.$$

Proof. In (2.6), set $a=aq^{-m}$ and shift the inner index of summation n to $n+m$ to obtain

$$\sum_{n=-m}^{\infty}\frac{(aq^{-m})_{n+m}}{(q)_{n+m}}z^{n+m}=\frac{(azq^{-m})_\infty}{(z)_\infty}. \tag{3.2}$$

Using the q-shifted factorial identity (2.5) in (3.2), we obtain

$$\sum_{n=-m}^{\infty}\frac{(a)_n}{(q^{m+1})_n}z^n=\frac{(az)_\infty(azq^{-m})_m(q)_m}{(z)_\infty(aq^{-m})_m z^m} = \frac{(az)_\infty(q/az)_m(q)_m}{(z)_\infty(q/a)_m}. \tag{3.3}$$

Letting $m\to\infty$ in (3.3), we obtain

$$\sum_{n=-\infty}^{\infty}(a)_n z^n=\frac{(az,q/az,q)_\infty}{(z,q/a)_\infty}. \tag{3.4}$$

The equation (3.4) can be written as

$$\sum_{n=0}^{\infty}(a)_n z^n - \frac{q}{az}\frac{1}{(1-q/a)}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(q^2/a)_n}\left(\frac{q}{az}\right)^n = \frac{(az,q/az,q)_\infty}{(z,q/a)_\infty}. \tag{3.5}$$

Putting $B=q$, $C=q^2/a$, $Z=q^2/(axz)$ in (2.7) and then letting $x\to\infty$, we obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(q^2/a)_n}\left(\frac{q}{az}\right)^n = \left(1-\frac{q}{a}\right)\sum_{n=0}^{\infty}\left(\frac{q}{z}\right)_n\left(\frac{q}{a}\right)^n. \tag{3.6}$$

Using (3.6) in (3.5), we obtain

$$\sum_{n=0}^{\infty}(a)_n z^n - \left(\frac{q}{az}\right)\sum_{n=0}^{\infty}\left(\frac{q}{z}\right)_n\left(\frac{q}{a}\right)^n = \frac{(az)_\infty(q/az)_\infty(q)_\infty}{(z)_\infty(q/a)_\infty}. \tag{3.7}$$

Replacing a by $-q/a$ and z by $-b$ in (3.7), we obtain (3.1), after some simplifications. This completes the proof. □

Theorem 3.2. *If* $|c|<|a|<1$ *and* $|c|<|b|<1$, *then*

$$f_3(a,b;c) - f_3(b,a;c) = \left(\frac{1}{a}-\frac{1}{b}\right)\frac{(q,aq/b,bq/a,c)_\infty}{(-a,-b,-c/a,-c/b)_\infty}, \tag{3.8}$$

where

$$f_3(a,b;c) = \frac{1}{a}\sum_{n=0}^{\infty}\frac{(-q/a)_n}{(-c/a)_{n+1}}(-b)^n.$$

Proof. Setting $A=q$, $B=-bq/d$, $C=c$, $D=-cq/a$ and $E=-bq$ in (2.8), we obtain

$$\sum_{n=0}^{\infty}\frac{(-bq/d,c)_n}{(-bq)_n(-c/a)_{n+1}}\left(\frac{-d}{a}\right)^n = (1+b)\sum_{n=0}^{\infty}\frac{(cd/ab,-q/a)_n}{(-c/a,-d/a)_{n+1}}(-b)^n. \tag{3.9}$$

Substituting (3.9) in (2.9), we obtain

$$\sum_{n=0}^{\infty}\frac{(c,d)_n}{(-aq,-bq)_n}q^n = -\frac{(c,d)_\infty}{a(-aq,-bq)_\infty}\sum_{n=0}^{\infty}\frac{(-bq/d)_n}{(-c/a)_{n+1}}(-d/a)^n + \left(1+\frac{1}{a}\right)(1+b)\sum_{n=0}^{\infty}\frac{(cd/ab,-q/a)_n}{(-c/a,-d/a)_{n+1}}(-b)^n. \tag{3.10}$$

Interchanging c and d in (3.10) and then setting $d=0$ in the resulting identity, we obtain

$$\sum_{n=0}^{\infty}\frac{(c)_n}{(-aq,-bq)_n}q^n = -\frac{(c)_\infty}{a(-aq,-bq)_\infty}\sum_{n=0}^{\infty}\left(\frac{-bq}{c}\right)_n\left(\frac{-c}{a}\right)^n + \left(1+\frac{1}{a}\right)(1+b)\sum_{n=0}^{\infty}\frac{(-q/a)_n}{(-c/a)_{n+1}}(-b)^n. \tag{3.11}$$

Interchanging a and b in (3.11), we obtain

$$\sum_{n=0}^{\infty}\frac{(c)_n}{(-aq,-bq)_n}q^n=-\frac{(c)_\infty}{b(-aq,-bq)_\infty}\sum_{n=0}^{\infty}\left(\frac{-aq}{c}\right)_n\left(\frac{-c}{b}\right)^n + \left(1+\frac{1}{b}\right)(1+a)\sum_{n=0}^{\infty}\frac{(-q/b)_n}{(-c/b)_{n+1}}(-a)^n. \quad (3.12)$$

Subtracting (3.12) from (3.11), we obtain after some simplifications

$$\frac{1}{a}\sum_{n=0}^{\infty}\frac{(-q/a)_n}{(-c/a)_{n+1}}(-b)^n-\frac{1}{b}\sum_{n=0}^{\infty}\frac{(-q/b)_n}{(-c/b)_{n+1}}(-a)^n = \frac{(c)_\infty}{(-a,-b)_\infty}\left[\frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{-bq}{c}\right)_n\left(\frac{-c}{a}\right)^n-\frac{1}{b}\sum_{n=0}^{\infty}\left(\frac{-aq}{c}\right)_n\left(\frac{-c}{b}\right)^n\right]. \quad (3.13)$$

Replacing a by c/b and b by c/a in (3.1) and then using the resulting identity in (3.13), we obtain (3.8) after some simplifications. This completes the proof. □

Theorem 3.3. *If* $|c|,\ |d|<|a|,\ |b|<1,$ *then*

$$f_4(a,b;c,d)-f_4(b,a;c,d) = \left(\frac{1}{a}-\frac{1}{b}\right)\frac{(q,aq/b,bq/a,c,d,cd/ab)_\infty}{(-a,-b,-c/a,-c/b,-d/a,-d/b)_\infty}, \quad (3.14)$$

where

$$f_4(a,b;c,d)=\frac{1}{a}\sum_{n=0}^{\infty}\frac{(-q/a,cd/ab)_n}{(-c/a,-d/a)_{n+1}}(-b)^n.$$

Proof. Interchange a and b in (2.9) to obtain

$$\sum_{n=0}^{\infty}\frac{(c,d)_n}{(-aq,-bq)_n}q^n=-\frac{(c,d)_\infty}{b(-aq,-bq)_\infty}\sum_{n=0}^{\infty}\frac{(-aq/d)_n}{(-c/b)_{n+1}}(-d/b)^n + \left(1+\frac{1}{b}\right)(1+a)\sum_{n=0}^{\infty}\frac{(cd/ab,-q/b)_n}{(-c/b,-d/b)_{n+1}}(-a)^n. \quad (3.15)$$

Subtract (3.15) from (3.10) to obtain

$$\frac{1}{a}\sum_{n=0}^{\infty}\frac{(-q/a,cd/ab)_n}{(-c/a,-d/a)_{n+1}}(-b)^n-\frac{1}{b}\sum_{n=0}^{\infty}\frac{(-q/b,cd/ab)_n}{(-c/b,-d/b)_{n+1}}(-a)^n = \frac{(c,d)_\infty}{(-a,-b)_\infty}\left[\frac{1}{a}\sum_{n=0}^{\infty}\frac{(-bq/c)_n}{(-d/a)_{n+1}}\left(\frac{-c}{a}\right)^n-\frac{1}{b}\sum_{n=0}^{\infty}\frac{(-aq/c)_n}{(-d/b)_{n+1}}\left(\frac{-c}{b}\right)^n\right]. \quad (3.16)$$

Replacing a by c/b and b by c/a in (3.8) and using the resulting identity in (3.16), we obtain (3.14) after some simplifications. This completes the proof. □

4. q-Gamma, q-Beta and Eta function identities

In this section, we deduce some interesting q-gamma, q-beta and eta function identities using symmetric forms of the reciprocity theorems.

Equation (3.1) can be written as

$$\sum_{n=0}^{\infty}\left(\frac{-q}{b}\right)_n(-a)^n-\frac{b}{a}\sum_{n=0}^{\infty}\left(\frac{-q}{a}\right)_n(-b)^n=\frac{(q,aq/b,b/a)_\infty}{(-a,-b)_\infty}. \tag{4.1}$$

Setting $a=-q^x$ and $b=-q^{x+y}$ in (4.1), we obtain

$$\sum_{n=0}^{\infty}\left(q^{1-x-y}\right)_n q^{xn}-q^y\sum_{n=0}^{\infty}\left(q^{1-x}\right)_n(q^{x+y})^n=\frac{(q,q^{1-y},q^y)_\infty}{(q^x,q^{x+y})_\infty}. \tag{4.2}$$

Using (2.1) in (4.2), we obtain after some simplifications

$$\frac{\Gamma_q(x)}{\Gamma_q(1-y)\Gamma_q(y)}=\frac{(q^{x+y})_\infty}{(1-q)^x(q)_\infty^2}\left[\sum_{n=0}^{\infty}\left(q^{1-x-y}\right)_n q^{nx}-q^y\sum_{n=0}^{\infty}\left(q^{1-x}\right)_n(q^{x+y})^n\right].$$

Similarly, setting $a=-q^x$ and $b=-q^{3x}$ in (4.1) and then using (2.1) in the resulting identity, we obtain after some simplifications

$$\frac{\Gamma_q(x)\Gamma_q(3x)}{\Gamma_q(2x)\Gamma_q(1-2x)}=\frac{(1-q)^{1-4x}}{(q)_\infty}\left[\sum_{n=0}^{\infty}\left(q^{1-3x}\right)_n q^{nx}-q^{2x}\sum_{n=0}^{\infty}\left(q^{1-x}\right)_n q^{3nx}\right].$$

Setting $a=-q^x$ and $b=-q^{1-y}$ in (4.1) and then using (2.1), we obtain after some simplifications

$$\Gamma_q(x)\Gamma_q(1-y)$$
$$=\frac{(1-q)^{1+y-x}(q)_\infty}{(q^{y+x},q^{1-y-x})_\infty}\left[\sum_{n=0}^{\infty}\left(q^{y}\right)_n q^{nx}-q^{1-y-x}\sum_{n=0}^{\infty}\left(q^{1-x}\right)_n(q^{1-y})^n\right].$$

Changing q to q^2 and then setting $a=-q$ and $b=q^2$ in (4.1) and then using (2.4), we obtain after some simplifications

$$\frac{\eta^7(2\tau)}{\eta^3(\tau)\eta^3(4\tau)}=q^{-1/24}\left[2\sum_{n=0}^{\infty}\left(-q^2;q^2\right)_{n-1}q^n+\sum_{n=0}^{\infty}(-1)^n(q;q^2)_nq^{2n+1}\right].$$

Changing q to q^2 and then setting $a=q$ and $b=-q$ in (4.1) and then using (2.4), we obtain after some simplifications

$$\frac{\eta^3(4\tau)}{\eta^2(2\tau)}=\frac{q^{1/3}}{2}\left[\sum_{n=0}^{\infty}(-1)^n(q;q^2)_nq^n+\sum_{n=0}^{\infty}\left(-q;q^2\right)_nq^n\right].$$

Setting $a=q^{1/2}$ and $b=q$ and then changing q to q^2 in (4.1) and then using (2.4), we obtain after some simplifications

$$\frac{\eta^3(\tau)}{\eta^2(2\tau)}=q^{-1/24}\left[2\sum_{n=0}^{\infty}(-1)^n(-q^2;q^2)_{n-1}q^n-\sum_{n=0}^{\infty}(-1)^n(-q;q^2)_nq^{2n+1}\right].$$

Changing q to q^2 and then setting $a = q$ and $b = -q^2$ in (4.1), we obtain after some simplifications

$$\chi(q) = \sum_{n=0}^{\infty} \left(-q; q^2\right)_n q^{2n+1},$$

where

$$\chi(q) = (-q; q^2)_\infty,$$

which can be found in Ramanujan's notebook [13, p.197].

Setting $a = q^{1/3}$ and $b = q^{2/3}$ and then changing q to q^3 in (4.1), we obtain after some simplifications

$$\frac{\eta^2(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)} = q^{1/8}\left[\sum_{n=0}^{\infty}(-1)^n(-q;q^3)_n q^n - \sum_{n=0}^{\infty}(-1)^n(-q^2;q^3)_n q^{2n+1}\right].$$

Equation (3.8) can be written as

$$\sum_{n=0}^{\infty} \frac{(-q/b)_n}{(-c/b)_{n+1}}(-a)^n - \frac{b}{a}\sum_{n=0}^{\infty} \frac{(-q/a)_n}{(-c/a)_{n+1}}(-b)^n = \frac{(q, aq/b, b/a, c)_\infty}{(-a, -b, -c/a, -c/b)_\infty}. \quad (4.3)$$

Setting $a = -q^x$, $b = -q^y$ and $c = q^{x+y}$ in (4.3), we obtain

$$\sum_{n=0}^{\infty} \frac{(q^{1-y})_n}{(q^x)_{n+1}} q^{xn} - q^{y-x}\sum_{n=0}^{\infty} \frac{(q^{1-x})_n}{(q^y)_{n+1}} q^{yn} = \frac{(q, q^{1+x-y}, q^{y-x}, q^{x+y})_\infty}{(q^x, q^y)^2_\infty}. \quad (4.4)$$

Using (2.1) and (2.3) in (4.4), we obtain after some simplifications

$$B_q^2(x,y) = \frac{(1-q)^2(q)_\infty(q^{x+y})_\infty}{(q^{1+x-y}, q^{y-x})_\infty}\left[\sum_{n=0}^{\infty} \frac{(q^{1-y})_n}{(q^x)_{n+1}} q^{xn} - q^{y-x}\sum_{n=0}^{\infty} \frac{(q^{1-x})_n}{(q^y)_{n+1}} q^{yn}\right].$$

Similarly, setting $a = -q^x$, $b = -q^y$, $c = q^{2x}$ in (4.3), using (2.1) and (2.3), we obtain after some simplifications

$$B_q(x,y) = \frac{(1-q)(q^x, q^{2x-y}, q^{x+y})_\infty}{(q^{1+x-y}, q^{y-x}, q^{2x})_\infty} \times \left[\sum_{n=0}^{\infty} \frac{(q^{1-y})_n}{(q^{2x-y})_{n+1}} q^{xn} - q^{y-x}\sum_{n=0}^{\infty} \frac{(q^{1-x})_n}{(q^x)_{n+1}} q^{yn}\right].$$

Setting $a = -q^x$, $b = -q^{1-x}$, $c = q^{2x}$ in (4.3) and then using (2.1), we obtain after some simplifications

$$\frac{\Gamma_q^2(x)\Gamma_q(1-x)\Gamma_q(3x-1)}{\Gamma_q^2(2x)\Gamma_q(1-2x)} = (1-q)^{2-2x}\left[\sum_{n=0}^{\infty} \frac{(q^x)_n}{(q^{3x-1})_{n+1}} q^{xn} - q^{1-2x}\sum_{n=0}^{\infty} \frac{(q^{1-x})_n}{(q^x)_{n+1}} (q^{1-x})^n\right].$$

Setting $a=-q^x,\ b=-q^y,\ c=q^{2x+y}$ in (4.3) and then using (2.1), we obtain after some simplifications

$$\Gamma_q(x)\Gamma_q(2x)\Gamma_q(y)=\frac{(1-q)^{3-3x-y}(q)_\infty^2(q^{x+y})_\infty}{(q^{1+x-y},q^{y-x},q^{2x+y})_\infty}\times\left[\sum_{n=0}^{\infty}\frac{(q^{1-y})_n}{(q^{2x})_{n+1}}q^{xn}-q^{y-x}\sum_{n=0}^{\infty}\frac{(q^{1-x})_n}{(q^{x+y})_{n+1}}q^{yn}\right].$$

Setting $a=-q^x,\ b=-q^{2x+y},\ c=q^{3x}$ in (4.3) and then using (2.1), we obtain after some simplifications

$$\frac{\Gamma_q(x)\Gamma_q(2x)}{\Gamma_q(3x)}=\frac{(1-q)(q^{2x+y},q^{x-y})_\infty}{(q^{1-x-y},q^{x+y})_\infty}\times\left[\sum_{n=0}^{\infty}\frac{(q^{1-2x-y})_n}{(q^{x-y})_{n+1}}q^{xn}-q^{x+y}\sum_{n=0}^{\infty}\frac{(q^{1-x})_n}{(q^{2x})_{n+1}}(q^{2x+y})^n\right].$$

Setting $a=-q^x,\ b=-q^{x+y},\ c=q^{2x}$ in (4.3) and then using (2.1), we obtain after some simplifications

$$\frac{\Gamma_q^2(x)}{\Gamma_q(2x)\Gamma_q(y)\Gamma_q(1-y)}=\frac{(q^{x+y},q^{x-y})_\infty}{(q)_\infty^2}\times\left[\sum_{n=0}^{\infty}\frac{(q^{1-x-y})_n}{(q^{x-y})_{n+1}}q^{xn}-q^{y}\sum_{n=0}^{\infty}\frac{(q^{1-x})_n}{(q^{x})_{n+1}}(q^{x+y})^n\right].$$

Setting $a=-q^x,\ b=-q^{2x},\ c=q^{3x}$ in (4.3) and then using (2.1), we obtain after some simplifications

$$\frac{\Gamma_q(x)\Gamma_q^2(2x)}{\Gamma_q(3x)\Gamma_q(1-x)}=(1-q)^{2-3x}\left[\sum_{n=0}^{\infty}\frac{(q^{1-2x})_n}{(q^{x})_{n+1}}q^{xn}-q^{x}\sum_{n=0}^{\infty}\frac{(q^{1-x})_n}{(q^{2x})_{n+1}}q^{2xn}\right].$$

Setting $a=-q^{1-x},\ b=-q^{2x-1},\ c=q^{2x}$ in (4.3) and then using (2.1), we obtain after some simplifications

$$\frac{\Gamma_q(1-x)}{\Gamma_q(3-3x)}=\frac{(1-q^{2x-1})(1-q)^{2-2x}}{(1-q^{3x-2})}\times\left[\sum_{n=0}^{\infty}\frac{(q^{2-2x})_n}{(q)_{n+1}}(q^{1-x})^n-q^{3x-2}\sum_{n=0}^{\infty}\frac{(q^{x})_n}{(q^{3x-1})_{n+1}}(q^{2x-1})^n\right].$$

Setting $a=-q^x,\ b=-q^{1-y},\ c=q^{1-x}$ in (4.3) and then using (2.1), we obtain after some simplifications

$$\frac{\Gamma_q(x)\Gamma_q(1-y)\Gamma_q(1-2x)}{\Gamma_q(1-x)}=\frac{(1-q)^{1+y}(q)_\infty(q^{y-x})_\infty}{(q^{x+y},q^{1-y-x})_\infty}\times\left[\sum_{n=0}^{\infty}\frac{(q^{y})_n}{(q^{y-x})_{n+1}}q^{nx}-q^{1-x-y}\sum_{n=0}^{\infty}\frac{(q^{1-x})_n}{(q^{1-2x})_{n+1}}(q^{1-y})^n\right].$$

Changing q to q^2 and then setting $a=-q,\ b=c=q^2$ in (4.3) and then using (2.4), we obtain after some simplifications

$$\phi^2(q)=2\left[\sum_{n=0}^{\infty}\frac{q^n}{(1+q^{2n})}+\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n}}{(1-q^{2n+1})}\right],$$

where

$$\phi(q)=\sum_{n=-\infty}^{\infty}q^{n^2}=\frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}$$

is one of the theta functions in Ramanujan's notebook [13, p.197].
Changing q to q^2 and then setting $a=q,\ b=c=-q^2$ in (4.3) and then using (2.4), we obtain after some simplifications

$$\frac{\eta^3(2\tau)}{\eta^2(\tau)\eta(4\tau)}=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-q;q^2)_n}{(q;q^2)_{n+1}}q^{2n+1}.$$

Changing q to q^2 and then setting $a=q,\ b=-q$ and $c=-q^2$ in (4.3) and then using (2.4), we obtain after some simplifications

$$\frac{\eta^4(4\tau)}{\eta^3(2\tau)}=\frac{q^{5/12}}{2}\left[\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^n}{(-q;q^2)_{n+1}}q^n+\sum_{n=0}^{\infty}\frac{(-q;q^2)_n}{(q;q^2)_{n+1}}q^n\right].$$

Changing q to q^2 and then setting $a=q,\ b=-q$ and $c=q^3$ in (4.3), we obtain after some simplifications

$$\frac{\eta^2(4\tau)\eta(\tau)}{\eta^3(2\tau)}=\frac{q^{1/8}}{2}\left[\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^n}{(q^2;q^2)_{n+1}}q^n+\sum_{n=0}^{\infty}\frac{(-q;q^2)_n}{(-q^2;q^2)_{n+1}}q^n\right].$$

Changing q to q^2 and then setting $a=-q^2,\ b=q^2$ and $c=-q^3$ in (4.3) and then using (2.4), we obtain after some simplifications

$$\frac{\eta(4\tau)}{\eta(\tau)}=q^{1/8}(1+q)\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_{n-1}}{(q;q^2)_{n+1}}q^{2n}.$$

Equation (3.14) can be written as

$$\sum_{n=0}^{\infty}\frac{(-q/b,cd/ab)_n}{(-c/b,-d/b)_{n+1}}(-a)^n-\frac{b}{a}\sum_{n=0}^{\infty}\frac{(-q/a,cd/ab)_n}{(-c/a,-d/a)_{n+1}}(-b)^n$$
$$=\frac{(q,aq/b,b/a,c,d,cd/ab)_\infty}{(-a,-b,-c/a,-c/b,-d/a,-d/b)_\infty}. \quad (4.5)$$

Setting $a=-q^{2x},\ b=-q^{2y},\ c=q^{3x}$ and $d=q^{3y}$ in (4.5), we obtain

$$\sum_{n=0}^{\infty}\frac{(q^{1-2y},q^{x+y})_n}{(q^{3x-2y},q^y)_{n+1}}q^{2xn}-q^{2y-2x}\sum_{n=0}^{\infty}\frac{(q^{1-2x},q^{x+y})_n}{(q^x,q^{3y-2x})_{n+1}}q^{2yn}$$
$$=\frac{(q,q^{1+2x-2y},q^{2y-2x},q^{3x},q^{3y},q^{x+y})_\infty}{(q^{2x},q^{2y},q^x,q^y,q^{3x-2y},q^{3y-2x})_\infty}. \quad (4.6)$$

Using (2.1) and (2.3) in (4.6), we obtain after some simplifications

$$\begin{aligned}\frac{B_q(x,y)B_q(2x,2y)}{B_q(3x,3y)} &= \frac{(1-q)(q^{3x-2y},q^{3y-2x},q^{2x+2y})_\infty}{(q^{1+2x-2y},q^{2y-2x},q^{3x+3y})_\infty}\\ &\times\left[\sum_{n=0}^{\infty}\frac{(q^{1-2y},q^{x+y})_n}{(q^{3x-2y},q^y)_{n+1}}q^{2xn}-q^{2y-2x}\sum_{n=0}^{\infty}\frac{(q^{1-2x},q^{x+y})_n}{(q^x,q^{3y-2x})_{n+1}}q^{2yn}\right].\end{aligned}$$

Similarly, setting $a=-q^x,\ b=-q^y,\ c=d=q^{2x}$ in (4.5) and then using (2.1) and (2.3), we obtain after some simplifications

$$\begin{aligned}B_q(x,y) &= \frac{(1-q)(q^{x+y})_\infty(q^{2x-y},q^x)^2_\infty}{(q^{1+x-y},q^{y-x},q^{3x-y})_\infty(q^{2x})^2_\infty}\\ &\times\left[\sum_{n=0}^{\infty}\frac{(q^{1-y},q^{3x-y})_n}{(q^{2x-y})^2_{n+1}}q^{xn}-q^{y-x}\sum_{n=0}^{\infty}\frac{(q^{1-x},q^{3x-y})_n}{(q^x)^2_{n+1}}q^{yn}\right].\end{aligned}$$

Setting $a=-q^x,\ b=-q^{1-y+x},\ c=d=q^{x+y}$ in (4.5) and then using (2.1) and (2.3), we obtain after some simplifications

$$\begin{aligned}\frac{B_q^2(x,y)}{B_q(x,1-y)} &= \frac{(1-q)(q^{2y-1})^2_\infty}{(q^y,q^{3y-1})_\infty}\\ &\times\left[\sum_{n=0}^{\infty}\frac{(q^{y-x},q^{3y-1})_n}{(q^{2y-1})^2_{n+1}}q^{xn}-q^{1-y}\sum_{n=0}^{\infty}\frac{(q^{1-x},q^{3y-1})_n}{(q^y)^2_{n+1}}(q^{1-y+x})^n\right].\end{aligned}$$

Setting $a=-q^x,\ b=-q^{1-x},\ c=d=q^{1+y}$ in (4.5) and then using (2.1), we obtain after some simplifications

$$\begin{aligned}\frac{\Gamma_q^3(x)\Gamma_q(1-x)}{B_q^2(x,y)\Gamma_q(2x)\Gamma_q(1-2x)\Gamma_q(1+2y)} &= (1-q^{1+y-x})^2_\infty(1-q^y)^2\\ &\times\left[\sum_{n=0}^{\infty}\frac{(q^x,q^{1+2y})_n}{(q^{x+y})^2_{n+1}}q^{xn}-q^{1-2x}\sum_{n=0}^{\infty}\frac{(q^{1-x},q^{1+2y})_n}{(q^{1+y-x})^2_{n+1}}(q^{1-x})^n\right].\end{aligned}$$

Setting $a=-q^{x-1},\ b=-q^{2x-1},\ c=d=q^{2x}$ in (4.5) and then using (2.1), we obtain after some simplifications

$$\begin{aligned}\frac{\Gamma_q(x+1)}{\Gamma_q(1-x)\Gamma_q(2x)} &= \frac{(1-q^{x-1})(1-q^{2x-1})(1-q^{x+1})}{(1-q)(q)_\infty}\\ &\times\left[\sum_{n=0}^{\infty}\frac{(q^{2-2x},q^{x+2})_n}{(q)^2_{n+1}}(q^{x-1})^n-q^x\sum_{n=0}^{\infty}\frac{(q^{2-x},q^{x+2})_n}{(q^{x+1})^2_{n+1}}(q^{2x-1})^n\right].\end{aligned}$$

Setting $a = -q^{1-2x},\ b = -q^{1-x},\ c = d = q^x$ in (4.5) and then using (2.1), we obtain after some simplifications

$$\frac{\Gamma_q(1-2x)\Gamma_q^2(3x-1)\Gamma_q^2(2x-1)}{\Gamma_q^3(x)\Gamma_q(5x-2)} = (1-q)^2 \times \left[\sum_{n=0}^{\infty} \frac{(q^x, q^{5x-2})_n}{(q^{2x-1})_{n+1}^2}(q^{1-2x})^n - q^x \sum_{n=0}^{\infty} \frac{(q^{2x}, q^{5x-2})_n}{(q^{3x-1})_{n+1}^2}(q^{1-x})^n\right].$$

Setting $a = -q^{1/2},\ b = q^{1/2},\ c = d = -q$ and then changing q to q^2 in (4.5) and then using (2.4), we obtain after some simplifications

$$\frac{\eta^8(4\tau)}{\eta^7(2\tau)} = \frac{q^{3/4}}{2}\left[\sum_{n=0}^{\infty} \frac{(-q, -q^2; q^2)_n}{(q; q^2)_{n+1}^2} q^n + \sum_{n=0}^{\infty} \frac{(-1)^n (q, -q^2; q^2)_n}{(-q; q^2)_{n+1}^2} q^n\right].$$

Setting $a = -q^{1/2},\ b = q^{1/2},\ c = d = q^{3/2}$ and then changing q to q^2 in (4.5) and then using (2.4), we obtain after some simplifications

$$\frac{1}{\phi(q)} = \frac{(1-q)^2}{2}\left[\sum_{n=0}^{\infty} \frac{(-q; q^2)_n}{(-q^2; q^2)_{n+1}} q^n + \sum_{n=0}^{\infty} \frac{(q; q^2)_n(-q^2; q^2)_{n+1}(-1)^n}{(q^2; q^2)_{n+1}^2} q^n\right].$$

Changing q to q^2 and then setting $a = -q^2,\ b = q^2,\ c = d = q^3$ in (4.5) and then using (2.4), we obtain after some simplifications

$$\frac{\eta^2(\tau)\eta^4(4\tau)}{\eta^6(2\tau)} = q^{1/4}(1-q)^2 \sum_{n=0}^{\infty} \frac{(-q^2; q^2)_{n-1}(-q^2; q^2)_n}{(-q; q^2)_{n+1}^2} q^{2n}.$$

Changing q to q^2 and then setting $a = q,\ b = -q^2,\ c = d = -q^3$ in (4.5) and then using (2.4), we obtain after some simplifications

$$\frac{\eta^2(2\tau)}{\eta^2(\tau)\eta^2(4\tau)} = q^{-1/4}(1+q)^2 \sum_{n=0}^{\infty} \frac{(-q; q^2)_n(-q; q^2)_{n+1}}{(q^2; q^2)_{n+1}^2} q^{2n+1}.$$

Changing q to q^2 and then setting $a = 1,\ b = q,\ c = d = q^2$ in (4.5) and then using (2.4), we obtain after some simplifications

$$\frac{\eta^6(\tau)}{\eta^3(2\tau)} = \frac{q^{3/4}}{2}\left[\sum_{n=0}^{\infty} \frac{(-q^2; q^2)_n}{(1+q^{2n+1})(-q; q^2)_{n+1}} q^n + \sum_{n=0}^{\infty} \frac{(-q^2; q^2)_n(-1)^n}{(1-q^{2n+1})(q; q^2)_{n+1}} q^n\right].$$

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