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COEFFICIENT DETERMINANTS INVOLVING MANY FEKETE-SZEGÖ-TYPE PARAMETERS

KUNLE OLADEJI BABALOLA

Abstract. We extend our definition (in a recent paper [3]) of the coefficient determinants of analytic mappings of the unit disk to include many Fekete-Szegö-type parameters, and compute the best possible bounds on certain specific determinants for the choice class of starlike functions.

1. Hankel Determinants with Parameters

In a recent paper [3], we extended the definition of the well known Hankel determinant for coefficients of analytic mappings to include the also well known Fekete-Szegö parameter as follows:

Definition 1.1. Let λ be a nonnegative real number. Then for integers $n \geq 1$ and $q \geq 1$, we define the q-th Hankel determinants with Fekete-Szegö parameter λ, that is $H_q^\lambda(n)$, as

$$H_q^\lambda(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & \lambda a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2(q-1)}. \end{vmatrix}$$

The well known Hankel determinant is the case $\lambda = 1$. We also defined other similar determinants as:

Definition 1.2. Let λ be a nonnegative real number. Then for integers $n \geq 1$ and $q \geq 1$, we define the $B_q^\lambda(n)$ determinants as

$$B_q^\lambda(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+q} & a_{n+q+1} & \cdots & a_{n+2q-1} \\ a_{n+2q} & a_{n+2q+1} & \cdots & a_{n+3q-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q(q-1)} & \cdots & \cdots & \lambda a_{n+q^2-1} \end{vmatrix}.$$

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Our motivation for those new definitions lies in the fact that, for function classes defined by other function classes (for example the classes of close-to-star, close-to-convex, quasi-convex, α-starlike, α-convex, α-close-to-star, α-close-to-convex whose definitions involve other function classes), coefficient functionals of the form $|a_2a_3 - \lambda a_4|$ and $|a_2a_4 - \lambda a_3^3|$ (and possibly more) for the defining function classes have frequently appeared to be resolved in the investigations of Hankel determinants for the desired classes of functions.

In this paper, we further extend these definitions to include finitely many Fekete-Szegö parameters λ_j, $j = 1, 2, \cdots$ in order to accomodate a wide variety of emerging functionals in the study of coefficients of mappings of the unit disk. Now we say:

Definition 1.3. Let λ_i, $i = 1, 2, \cdots, q$ be nonnegative real numbers. Then for integers $n \geq 1$ and $q \geq 1$, we define the q-th Hankel determinants with Fekete-Szegö parameters λ_i, that is $H_q^{\lambda_1,\lambda_2,\cdots,\lambda_q}(n)$, as

$$H_q^{\lambda_1,\lambda_2,\cdots,\lambda_q}(n) = \begin{vmatrix} \lambda_1 a_n & \lambda_2 a_{n+1} & \cdots & \lambda_q a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2(q-1)} \end{vmatrix}$$

and we also say:

Definition 1.4. Let λ_j, $j = 1, 2, \cdots, n$ be nonnegative real numbers. Then for integers $n \geq 1$ and $q \geq 1$, we define the $B_q^{\lambda_1,\lambda_2,\cdots,\lambda_n}(n)$ determinants as

$$B_q^{\lambda_1,\lambda_2,\cdots,\lambda_n}(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & \lambda_1 a_{n+q-1} \\ a_{n+q} & a_{n+q+1} & \cdots & \lambda_2 a_{n+2q-1} \\ a_{n+2q} & a_{n+2q+1} & \cdots & \lambda_3 a_{n+3q-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q(q-1)} & \cdots & \cdots & \lambda_n a_{n+q^2-1} \end{vmatrix}.$$

For $\lambda_j = 1$, $j = 1, 2, \cdots, q-1(n-1)$, we simply write $H_q^{\lambda}(n)$ and $B_q^{\lambda}(n)$ in place of $H_q^{1,1,\cdots,1,\lambda_q}(n)$ and $B_q^{1,1,\cdots,1,\lambda_n}(n)$ respectively, and we quickly note that for real numbers, γ, α and β we have:

$$H_2^{\gamma}(1) = \begin{vmatrix} 1 & \gamma a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - \gamma a_2^2,$$

$$H_2^{\alpha}(2) = \begin{vmatrix} a_2 & \alpha a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - \alpha a_3^2$$

and

$$B_2^{\beta}(1) = \begin{vmatrix} 1 & a_2 \\ a_3 & \beta a_4 \end{vmatrix} = a_2a_3 - \beta a_4.$$

Then, in this paper we shall investigate the determinant $H_3^{\lambda_1,\lambda_2,\lambda_3}(1)$ for the favoured and well known class of starlike functions, for which Re $zf'(z)/f(z) > 0$, denoted by S^*. By definition

$$H_3^{\lambda_1,\lambda_2,\lambda_3}(1) = \begin{vmatrix} \lambda_1 a_1 & \lambda_2 a_2 & \lambda_3 a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

For $f \in S^*$, $a_1 = 1$ so that

$$\begin{aligned} H_3^{\lambda_1,\lambda_2,\lambda_3}(1) &= a_3(\lambda_2 a_2 a_4 - \lambda_3 a_3^2) - a_4(\lambda_1 a_4 - \lambda_3 a_2 a_3) + a_5(\lambda_1 a_3 - \lambda_2 a_2^2) \\ &= \lambda_2 a_3(a_2 a_4 - \alpha a_3^2) + \lambda_3 a_4(a_2 a_3 - \beta a_4) + \lambda_1 a_5(a_3 - \gamma a_2^2) \end{aligned}$$

where $\alpha = \lambda_3/\lambda_2$, $\beta = \lambda_1/\lambda_3$ and $\gamma = \lambda_2/\lambda_1$. Thus by triangle inequality, we have

$$|H_3^{\lambda_1,\lambda_2,\lambda_3}(1)| \le \lambda_1|a_5||H_2^{\gamma}(1)| + \lambda_2|a_3||H_2^{\alpha}(2)| + \lambda_3|a_4||B_2^{\beta}(1)|. \tag{1.1}$$

Keen attention is paid to the dynamics of the Fekete-Szegö parameters in this paper, which leads to ease of analysis and better precision. For our choice class, S^*, the Fekete-Szegö functional, $|H_2^{\gamma}(1)|$, is known and given as:

Theorem 1.5 ([6])**.** *Let $f \in S^*$. Then for real number γ,*

$$|H_2^{\gamma}(1)| \le \max\{1, |4\gamma - 3|\}.$$

The inequality is sharp. For each γ, equality is attained by $f(z)$ given by

$$f(z) = \begin{cases} \frac{z}{1-z^2} & if \quad \frac{1}{2} \le \gamma \le 1, \\ \frac{z}{(1-z)^2} & if \quad \gamma \in \left[0, \frac{1}{2}\right] \cup [1, \infty). \end{cases}$$

Thus we shall in this paper obtain the best possible bounds on $H_2^{\alpha}(2)$ and $B_2^{\beta}(1)$ to conclude our investigation. In the next section we state the lemmas we shall use to establish the desired bounds in Section 3.

2. Preliminary Lemmas

Let P denote the class of functions $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ which are regular in E and satisfy Re $p(z) > 0$, $z \in E$. To prove the main results in the next section we shall require the following lemmas.

Lemma 2.1 ([4])**.** *Let $p \in P$, then $|c_k| \le 2$, $k = 1, 2, \cdots$, and the inequality is sharp. Equality is realized by the Möbius function $L_0(z) = (1+z)/(1-z)$.*

Lemma 2.2 ([1])**.** *Let $p \in P$. Then*

$$\left|c_2 - \sigma\frac{c_1^2}{2}\right| \le \begin{cases} 2(1-\sigma), & \sigma \le 0, \\ 2, & 0 \le \sigma \le 2, \\ 2(\sigma - 1), & \sigma \ge 2 \end{cases} = 2\max\{1, |\sigma - 1|\}.$$

The inequality is sharp. For each σ, equality is attained by $p(z)$ given by

$$p(z) = \begin{cases} \frac{1+z^2}{1-z^2} & if \quad 0 \le \sigma \le 2, \\ \frac{1+z}{1-z} & if \quad \gamma \in [-\infty, 0] \cup [2, \infty). \end{cases}$$

Lemma 2.3 ([7]). *Let $p \in P$, then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{2.1}$$

and

$$4c_3 = c_1^3 + 2xc_1(4 - c_1^2) - x^2c_1(4 - c_1^2) + 2z(1 - |x|^2)(4 - c_1^2) \tag{2.2}$$

for some x, z such that $|x| \le 1$ and $|z| \le 1$.

3. Inequalities for $|H_3^{\lambda_1,\lambda_2,\lambda_3}(1)|$ of S^*

First we prove:

Theorem 3.1. *Let $f \in S^*$. Then for real number β,*

$$|B_2^\beta(1)| \le 2\max\{\beta, |3 - 2\beta|\}.$$

The inequalities are sharp. For each β, equality is attained by $f(z)$ given by

$$f(z) = \begin{cases} \frac{z}{(1-z^3)^2} & \text{if} \quad 1 \le \beta \le 3, \\ \frac{z}{(1-z)^2} & \text{if} \quad \beta \in [0,1] \cup [3, \infty). \end{cases}$$

Proof. It is well known that if $f \in S^*$, then $a_2 = c_1$, $2a_3 = c_2 + c_1^2$ and $6a_4 = 2c_3 + 3c_1c_2 + c_1^3$. Thus we have

$$|B_2^\beta(1)| = |a_2a_3 - \beta a_4| = \left|\frac{3-\beta}{6}c_1^3 + \frac{1-\beta}{2}c_1c_2 - \beta\frac{c_3}{3}\right|. \tag{3.1}$$

First observe that if $\beta \le 1$, then (3.1) yields

$$|B_2^\beta(1)| \le \left|\frac{3-\beta}{6}c_1^3 - \beta\frac{c_3}{3}\right| + \left|\frac{1-\beta}{2}c_1c_2\right|$$

which, by Lemma 2.1 gives

$$|B_2^\beta(1)| \le \left|\frac{3-\beta}{6}c_1^3 - \beta\frac{c_3}{3}\right| + 2(1-\beta). \tag{3.2}$$

Now substituting for c_3 using Lemma 2.3, we have

$$\begin{aligned}\left|\frac{3-\beta}{6}c_1^3 - \beta\frac{c_3}{3}\right| = \bigg|&\frac{(2-\beta)c_1^3}{4} - \frac{\beta c_1(4-c_1^2)x}{6} + \frac{\beta c_1(4-c_1^2)^2x^2}{12} \\ &-\frac{\beta(4-c_1^2)(1-|x|^2)z}{6}\bigg|.\end{aligned}$$

By Lemma 2.1 again, $|c_1| \le 2$. Then letting $c_1 = c$, we may assume without restriction that $c \in [0, 2]$ so that

$$\begin{aligned}\left|\frac{3-\beta}{6}c_1^3 - \beta\frac{c_3}{3}\right| &\le \frac{(2-\beta)c^3}{4} + \frac{\beta(4-c^2)}{6} + \frac{\beta c(4-c^2)\rho}{6} + \frac{\beta(c-2)(4-c^2)\rho^2}{12} \\ &= F(\rho).\end{aligned}$$

Since $\beta \le 1$, the extreme points of $F(\rho)$ are $\rho = 0$, $\rho = 1$ and $\rho = c/(2-c)$ (with $c \in [0, 1]$ in this case since $\rho \in [0, 1]$). Now let

$$G_1(c) = F(0) = \frac{(2-\beta)c^3}{4} + \frac{\beta(4-c^2)}{6}$$

$$G_2(c) = F(1) = \frac{(2-\beta)c^3}{4} + \frac{\beta(4-c^2)}{6} + \frac{\beta c(4-c^2)}{6} + \frac{\beta(c-2)(4-c^2)}{12}$$
$$= \frac{(1-\beta)c^3}{2} + \beta c$$

and

$$G_3(c) = F\left(\frac{c}{2-c}\right) = \frac{(2-\beta)c^3}{4} + \frac{\beta(4-c^2)}{6} + \frac{\beta c^2(c+2)}{12}$$
$$= \frac{(3-\beta)c^3}{6} + \frac{2\beta}{3}, \quad c \in [0,1].$$

By elementary calculus, we find that $G_1(c) \leq G_2(c) \leq G_2(2) = 4 - 2\beta$ while $G_3(c) \leq G_3(1) = (1+\beta)/2$. Hence for $\beta \leq 1$ the maximum of the functional is $4 - 2\beta$. Using this in (3.2) we have $|B_2^\beta(1)| \leq 6 - 4\beta$ for $0 \leq \beta \leq 1$.

Next we consider the case $1 \leq \beta \leq 3$. Then we write (3.1)

$$|B_2^\beta(1)| = \left|\beta\frac{c_3}{3} + \frac{\beta-1}{2}c_1c_2 - \frac{\beta-3}{6}c_1^3\right|$$

which gives

$$|B_2^\beta(1)| \leq \beta\frac{|c_3|}{3} + \frac{\beta-1}{2}|c_1|\left|c_2 - \frac{\beta-3}{3(\beta-1)}\frac{c_1^2}{2}\right|.$$

Observing that $(\beta-3)/(3\beta-3) \leq 0$ and applying Lemmas 2.1 and 2.2, we have $|B_2^\beta(1)| \leq 2\beta$ for $1 \leq \beta \leq 3$.

Finally, if $\beta \geq 3$ we write (3.1) as

$$|B_2^\beta(1)| = \left|\beta\frac{c_3}{3} + \frac{\beta-1}{2}c_1c_2 + \frac{\beta-3}{6}c_1^3\right|$$

and apply Lemma 2.1 to obtain $|B_2^\beta(1)| \leq 4\beta - 6$ in this case. Hence we have

$$|B_2^\beta(1)| \leq \begin{cases} 6-4\beta & if \quad 0 \leq \beta \leq 1, \\ 2\beta & if \quad 1 \leq \beta \leq 3, \\ 4\beta - 6 & if \quad \beta \geq 3. \end{cases}$$

This completes the proof. □

Corollary 3.2 ([2]). *Let $f \in S^*$. Then*

$$|B_2(1)| \leq 2.$$

The inequality is sharp. Equality is attained by the Koebe function $k(z) = z/(1-z)^2$.

Next we have

Theorem 3.3. *Let $f \in S^*$. Then for real number α,*

$$|H_2^\alpha(2)| \leq \max\{1, |9\alpha - 8|\}.$$

The inequalities are sharp. For each α, equality is attained by $f(z)$ given by

$$f(z)=\begin{cases}\frac{z}{1-z^2/\sqrt{\alpha}} & \textit{if} \quad \frac{2}{3}<\alpha\leq 1,\\ \frac{z}{(1-z)^2} & \textit{if} \quad \alpha\in\left[0,\frac{2}{3}\right]\cup[1,\infty).\end{cases}$$

Proof. Following from the proof of Theorem 3.1, if $f\in S^*$, then $a_2=c_1$, $2a_3=c_2+c_1^2$ and $6a_4=2c_3+3c_1c_2+c_1^3$. Thus we have

$$|H_2^{\alpha}(2)|=|a_2a_4-\alpha a_3^2|=\left|\frac{c_1c_3}{3}+(1-\alpha)\frac{c_1^2c_2}{2}+(2-3\alpha)\frac{c_1^4}{12}-\alpha\frac{c_2^2}{4}\right|. \tag{3.3}$$

First we suppose $\alpha\leq 2/3$. Then we have

$$|H_2^{\alpha}(2)|\leq\frac{|c_1||c_3|}{3}+(2-3\alpha)\frac{|c_1|^4}{12}+\frac{\alpha|c_2|}{4}\left|c_2-\frac{4(1-\alpha)}{\alpha}\frac{c_1^2}{2}\right|$$

which, by Lemmas 2.1 and 2.2 noting that $4(1-\alpha)/\alpha\geq 2$ for $\alpha\leq 2/3$, gives $|H_2^{\alpha}(2)|\leq 8-9\alpha$.

Substituting for c_2 and c_3 in (3.3) using Lemma 2.3 we obtain

$$\begin{aligned}|H_2^{\alpha}(2)|=\bigg|&\frac{(8-9\alpha)c_1^4}{16}+\frac{(10-9\alpha)c_1^2(4-c_1^2)x}{24}-\frac{c_1^2(4-c_1^2)x^2}{12}\\ &+\frac{c_1(1-|x|^2)(4-c_1^2)z}{6}-\frac{\alpha(4-c_1^2)x^2}{16}\bigg|\end{aligned} \tag{3.4}$$

Next we write (3.4) as:

$$\begin{aligned}|H_2^{\alpha}(2)|=\bigg|&\frac{(9\alpha-8)c_1^4}{16}-\frac{(10-9\alpha)c_1^2(4-c_1^2)x}{24}+\frac{c_1^2(4-c_1^2)x^2}{12}\\ &-\frac{c_1(1-|x|^2)(4-c_1^2)z}{6}+\frac{\alpha(4-c_1^2)x^2}{16}\bigg|.\end{aligned}$$

Letting $c_1=c$, assuming without restriction that $c\in[0,2]$ and setting $\rho=|x|$, we have

$$\begin{aligned}|H_2^{\alpha}(2)|\leq&\frac{(9\alpha-8)c^4}{16}+\frac{c(4-c^2)}{6}+\frac{(10-9\alpha)c^2(4-c^2)\rho}{24}\\ &\frac{(4-c^2)(c-2)[(4-3\alpha)c-6\alpha]\rho^2}{48}=F(\rho).\end{aligned}$$

Then

$$F'(\rho)=\frac{(10-9\alpha)c^2(4-c^2)}{24}+\frac{(4-c^2)(c-2)[(4-3\alpha)c-6\alpha]\rho}{24}$$

which is positive whenever $2/3\leq\alpha\leq 10/9$. Hence for $2/3\leq\alpha\leq 10/9$, $F(\rho)$ is increasing on $[0,1]$ so that $F(\rho)\leq F(1)$. Thus we have

$$|H_2^{\alpha}(2)|\leq F(1)=(\alpha-1)c^4-2(\alpha-1)c^2+\alpha=G(c)$$

Now by elementary calculus we see that the maximum of $G(c)$ on $[0,2]$ occurs at $c=1$ if $2/3\leq\alpha\leq 1$ while it is at $c=2$ for $1\leq\alpha\leq 10/9$ and is thus given by $G(1)=1$ for $2/3\leq\alpha\leq 1$ and $G(2)=9\alpha-8$ for $1\leq\alpha\leq 10/9$.

Next suppose $\alpha \geq 10/9$. Then we write (3.4) as

$$|H_2^\alpha(2)| = \left| \frac{(9\alpha-8)c_1^4}{16} + \frac{(9\alpha-10)c_1^2(4-c_1^2)x}{24} + \frac{c_1^2(4-c_1^2)x^2}{12} \right. \\ \left. - \frac{c_1(1-|x|^2)(4-c_1^2)z}{6} + \frac{\alpha(4-c_1^2)x^2}{16} \right|.$$

By similar argument as above we have

$$|H_2^\alpha(2)| \leq F(1) = \frac{3\alpha-2}{12}c^4 + \frac{3\alpha-4}{3}c^2 + \alpha = G(c)$$

which yields $|H_2^\alpha(2)| \leq 9\alpha-8$ similarly. Hence for $\alpha \geq 1$ we have $|H_2^\alpha(2)| \leq 9\alpha-8$.

We thus conclude that

$$|H_2^\alpha(2)| \leq \begin{cases} 8-9\alpha & if \quad 0 \leq \alpha \leq \frac{2}{3}, \\ 1 & if \quad \frac{2}{3} < \alpha \leq 1, \\ 9\alpha-8 & if \quad \alpha \geq 1 \end{cases}$$

which completes the proof. □

Corollary 3.4 ([5]). *Let $f \in S^*$. Then*

$$|H_2(2)| \leq 1.$$

The inequality is sharp. Equality is attained by the Koebe function $k(z) = z/(1-z)^2$.

Now using Theorems 1.5, 3.1 and 3.3 in (1.1) and setting $\alpha = \lambda_3/\lambda_2$, $\beta = \lambda_1/\lambda_3$ and $\gamma = \lambda_2/\lambda_1$, we obtain the grand inequalities for $|H_3^{\lambda_1,\lambda_2,\lambda_3}(1)|$ as follows:

Theorem 3.5. *Let $f \in S^*$. Then for real numbers λ_j, $j = 1, 2, 3$,*

$$|H_3^{\lambda_1,\lambda_2,\lambda_3}(1)| \leq 5\max\{\lambda_1, |4\lambda_2 - 3\lambda_1|\} + 8\max\{\lambda_1, |3\lambda_3 - 2\lambda_1|\} \\ + 3\max\{\lambda_2, |9\lambda_3 - 8\lambda_2|\}.$$

The inequalities are sharp.

We conclude with the following interesting and nice bounds on coefficient determinants of starlike functions.

Corollary 3.6. *Let $f \in S^*$. Then*

$$|H_3^{1,1,1}(1)| \leq 16,\ |H_3^{1,1,2}(1)| \leq 81,\ |H_3^{1,2,1}(1)| \leq 51.5,\ |H_3^{2,1,1}(1)| \leq 29,$$

$$|H_3^{1,2,2}(1)| \leq 63,\ |H_3^{2,1,2}(1)| \leq 78,\ |H_3^{2,2,1}(1)| \leq 52.5,\ |H_3^{1,3,2}(1)| \leq 105.$$

All inequalities are best possible in the sense that component inequalities in the last theorem are best possible.

The first $|H_3(1)| = |H_3^{1,1,1}(1)| \leq 16$ was reported in [2].

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Current: Department of Physical Sciences, Al-Hikmah University, Ilorin, Nigeria.

Permanent: Department of Mathematics, University of Ilorin, Ilorin, Nigeria.

E-mail address: kobabalola@gmail.com, babalola.ko@unilorin.edu.ng