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ON GENERALIZED HIGHER DERIVATIONS ON SEMIPRIME RINGS

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This paper is dedicated to Professor M. N. Daif

Abstract. We obtain the following result: Let R be a 2-torsion free semiprime $*-$ring. Suppose there exists a family of additive mappings $F=(f_i)_{i\in\mathbb{N}_0}$ of R associated with some higher derivation $D=(d_i)_{i\in\mathbb{N}_0}$ of R, such that $f_0=id_R$ and the relation $f_n(xx^*)=\sum_{i+j=n} f_i(x)d_j(x^*)$ is fulfilled for each $n\in\mathbb{N}_0$ and for all $x\in R$. Then F is a generalized higher derivation. Further, we also prove that if the relation $f_n(xy^*x)=\sum_{i+j+k=n} f_i(x)d_j(y^*)d_k(x)$ is fulfilled for each $n\in\mathbb{N}_0$ and for all $x,y\in R$, then F is also a generalized higher derivation.

1. Introduction

Throughout, R will represent an associative ring with center $Z(R)$. Recall that R is *prime* if $xRy=\{0\}$ implies $x=0$ or $y=0$, and is *semiprime* if $xRx=\{0\}$ implies $x=0$. An additive mapping $x\to x^*$ satisfying $(xy)^*=y^*x^*$ and $(x^*)^*=x$ for all $x,y\in R$ is called an *involution* and R is called a $*-$ring. An additive mapping $d:R\to R$ is called a *derivation* (resp. a *Jordan derivation*) if $d(xy)=d(x)y+xd(y)$ (resp. $d(x^2)=d(x)x+xd(x)$) holds for all $x,y\in R$. Clearly, every derivation is a Jordan derivation; in general the converse is not true, see [6, Section 3]. In the class of prime rings with characteristic not two, Herstein [13] has proved his classical theorem which shows that any Jordan derivation is a derivation. Later on, Herstein's theorem has been extended by Cusack [6] to 2-torsion free semiprime rings. An additive mapping $f:R\to R$ is said to be a *generalized derivation* (resp. *generalized Jordan derivation*) if there exists a derivation (resp. Jordan derivation) $d:R\to R$ such that $f(xy)=f(x)y+xd(y)$ (resp. $f(x^2)=f(x)x+xd(x)$) holds for all $x,y\in R$. Generalized derivations have been primarily defined by Brešar [4]. In fact, the concept of generalized derivations (resp. generalized Jordan derivations) covers that of derivations (resp. Jordan derivations). Moreover, generalized derivations (resp. generalized Jordan

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derivations) with $d=0$ covers the concept of left centralizers (resp. Jordan left centralizers), that is, an additive map T satisfying $T(xy)=T(x)y$ (resp. $T(x^2)=T(x)x$), for all $x,y\in R$. This line of investigations has widely been studied and several interesting results have been obtained. See [1] for some recent works and further references on derivations.

Following Hasse and Schmidt [12], let $\mathbb{N}_0$ be the set of all nonnegative integers and $D=(d_i)_{i\in\mathbb{N}_0}$ be a family of additive mappings of a ring R such that $d_0=id_R$. Then D is said to be a *higher derivation*, (resp. a *Jordan higher derivation*) of R if for each $n\in\mathbb{N}_0, d_n(xy)=\sum_{i+j=n}d_i(x)d_j(y)$ (resp. $d_n(x^2)=\sum_{i+j=n}d_i(x)d_j(x)$) holds for all $x,y\in R$. Algebraists sometimes called this notion of higher derivations *Hasse–Schmidt derivations*. It is easy to see that the first member of each higher derivation is itself a derivation. Ferrero and Haetinger [10] have extended Herstein's theorem [13] for higher derivations on 2-torsion free semiprime rings. Later on, Cortes and Haetinger [5] have defined the concept of generalized higher derivations. A family $F=(f_i)_{i\in\mathbb{N}_0}$ of additive mappings of a ring R such that $f_0=id_R$ is said to be a *generalized higher derivation*, (resp. a *generalized Jordan higher derivation*) of R if there exists a higher derivation (resp. Jordan higher derivation) $D=(d_i)_{i\in\mathbb{N}_0}$ and for each $n\in\mathbb{N}_0, f_n(xy)=\sum_{i+j=n}f_i(x)d_j(y)$ (resp. $f_n(x^2)=\sum_{i+j=n}f_i(x)d_j(x)$) holds for all $x,y\in R$. Obviously, every generalized higher derivation is a generalized Jordan higher derivation but the converse need not be true. It has been proved in [5] that if R is a 2-torsion free ring which has a commutator right nonzero divisor and U is a square closed Lie ideal of R, then every generalized higher derivation of U into R is a generalized higher derivation of U into R. Wei and Xiao [15] have proved that every generalized Jordan higher derivation on a 2-torsion free semiprime ring is a generalized higher derivation. For an account on higher derivations we refer the reader to [11] and to some of the most recent article [2, 3, 8, 9].

In [7], Daif and El-Sayiad have proved the following result Let R be a 2-torsion free semiprime $*-$ring and $F:R\to R$ an additive mapping associated with a derivation $d:R\to R$ such that $F(xx^*)=F(x)x^*+xd(x^*)$ holds for all $x\in R$. Then F is a generalized Jordan derivation. In the present work our main goal is to extend this result for generalized Jordan higher derivations. In addition, we show that if the relation $f_n(xy^*x)=\sum_{i+j+k=n}f_i(x)d_j(y^*)d_k(x)$ is fulfilled for each $n\in\mathbb{N}_0$ and for all $x,y\in R$, then F is a generalized higher derivation.

2. Main results

We shall need the following two results in the proof of our theorems.

Lemma 2.1 ([14, Lemma 1])**.** *Let R be a semiprime $*-$ring. Suppose there exists an element $a\in R$ such that $ax^*=ax$ is fulfilled for all $x\in R$. In this case $a\in Z(R)$.*

Lemma 2.2 ([15, Corollary 2.17])**.** *Let R be a 2-torsion free semiprime ring. Then every generalized Jordan higher derivation of R is a generalized higher derivation of R.*

Theorem 2.3. *Let R be a 2-torsion free semiprime $*-$ring. Suppose there exists a family of additive mappings $F = (f_i)_{i\in\mathbb{N}_0}$ of R associated with some higher derivation $D = (d_i)_{i\in\mathbb{N}_0}$ of R such that $f_0 = id_R$, and the relation $f_n(xx^*) = \sum_{i+j=n} f_i(x)d_j(x^*)$ is fulfilled for each $n \in \mathbb{N}_0$ and for all $x \in R$. Then F is a generalized higher derivation.*

Proof. Our hypothesis is

$$f_n(xx^*) = \sum_{i+j=n} f_i(x)d_j(x^*), \quad \text{for all } x \in R. \tag{2.1}$$

Linearizing (2.1) gives for all $x, y \in R$ that

$$f_n(xy^* + yx^*) = \sum_{i+j=n} f_i(x)d_j(y^*) + f_i(y)d_j(x^*). \tag{2.2}$$

Putting $y = x^*$ in (2.2) we get

$$A_n(x) + A_n(x^*) = 0, \text{ for all } x \in R, \tag{2.3}$$

where $A_n(x)$ denotes $f_n(x^2) - \sum_{i+j=n} f_i(x)d_j(x)$.

We proceed by induction to show that $A_n(x) = 0$ for all $x \in R$ and for each positive integer n. If $n = 0$, it is easy to obtain $A_0(x) = 0$ for all $x \in R$. When $n = 1$, by [7, Theorem 2.1.] we know that $A_1(x) = 0$ for all $x \in R$. We now suppose in all the next steps that $A_m(x) = 0$ for all $x \in R$ and each positive integer $m < n$.

For any $x, y \in R$, let us set $w = f_n(x(xy^* + yx^*)^* + (xy^* + yx^*)x^*)$. Then using (2.2) we get

$$\begin{aligned}
w &= \sum_{i+j=n} f_i(x)d_j(xy^* + yx^*) + f_i(xy^* + yx^*)d_j(x^*) \\
&= \sum_{i+j=n} f_i(x) \sum_{p+q=j} d_p(x)d_q(y^*) + \sum_{i+j=n} f_i(x) \sum_{p+q=j} d_p(y)d_q(x^*) \\
&\quad + \sum_{i+j=n} \sum_{s+t=i} f_s(x)d_t(y^*)d_j(x^*) + \sum_{i+j=n} \sum_{s+t=i} f_s(y)d_t(x^*)d_j(x^*) \\
&= \sum_{i+p=n} f_i(x)d_p(x)y^* + \sum_{\substack{i+p+q=n \\ i+p\neq n}} f_i(x)d_p(x)d_q(y^*) \\
&\quad + \sum_{i+p+q=n} f_i(x)d_p(y)d_q(x^*) + \sum_{s+t+j=n} f_s(x)d_t(y^*)d_j(x^*) \\
&\quad + \sum_{s+t+j=n} f_s(y)d_t(x^*)d_j(x^*).
\end{aligned}$$

On the other hand we can write w as follows

$$\begin{aligned}
w &= f_n(x(y+y^*)x^*) + f_n(x^2y^* + yx^{2^*}) \\
&= f_n(x(y+y^*)x^*) + \sum_{i+j=n} f_i(x^2)d_j(y^*) + \sum_{i+j=n} f_i(y)d_j(x^{2^*}) \\
&= f_n(x(y+y^*)x^*) + f_n(x^2)y^* + \sum_{\substack{p+q+j=n \\ p+q\neq n}} f_p(x)d_q(x)d_j(y^*) \\
&\quad + \sum_{i+s+t=n} f_i(y)d_s(x^*)d_t(x^*).
\end{aligned}$$

Comparing the last two forms of w we get for all $x, y \in R$ that

$$f_n(x(y+y^*)x^*) = -A_n(x)y^* + \sum_{i+j+k=n} f_i(x)d_j(y+y^*)d_k(x^*). \tag{2.4}$$

Putting $y - y^*$ for y in (2.4) we obtain

$$A_n(x)y = A_n(x)y^* \text{ for all } x, y \in R. \tag{2.5}$$

From Lemma 2.1, (2.5) gives $A_n(x) \in Z(R)$ for all $x \in R$. Replacing y by y^* in (2.2) we obtain for all $x, y \in R$

$$f_n(xy + y^*x^*) = \sum_{i+j=n} f_i(x)d_j(y) + f_i(y^*)d_j(x^*). \tag{2.6}$$

Putting xy for y in (2.6) gives for all $x, y \in R$

$$\begin{aligned}
f_n(x^2y + y^*x^{2^*}) &= \sum_{i+j=n} f_i(x)d_j(xy) + f_i(y^*x^*)d_j(x^*) \\
&= \sum_{\substack{i+p+q=n \\ i+p\neq n}} f_i(x)d_p(x)d_q(y) + \sum_{i+p=n} f_i(x)d_p(x)y \\
&\quad + \sum_{\substack{i+j=n \\ i\neq n}} f_i(y^*x^*)d_j(x^*) + f_n(y^*x^*)x^*.
\end{aligned}$$

On the other hand, the substitution x^2 for x in (2.6) gives for all $x, y \in R$

$$\begin{aligned}
f_n(x^2y + y^*x^{2^*}) &= \sum_{i+j=n} f_i(x^2)d_j(y) + f_i(y^*)d_j(x^{2^*}) \\
&= f_n(x^2)y + \sum_{\substack{s+t+j=n \\ s+t\neq n}} f_s(x)d_t(x)d_j(y) \\
&\quad + \sum_{\substack{i+p+q=n \\ i+p\neq n}} f_i(y^*)d_p(x^*)d_q(x^*) + \sum_{i+p=n} f_i(y^*)d_p(x^*)x^*.
\end{aligned}$$

Comparing the last two equations so obtained we have

$$A_n(x)y+\Big(-f_n(y^*x^*)+\sum_{i+p=n} f_i(y^*x^*)\Big)x^* + \sum_{\substack{i+p+q=n\\ i+p\neq n}} f_i(y^*)d_p(x^*)d_q(x^*) - \sum_{\substack{i+j=n\\ i\neq n}} f_i(y^*x^*)d_j(x^*) = 0. \tag{2.7}$$

In particular for $y=x$ and using our induction assumption the relation (2.7) reduces to

$$A_n(x)x - A_n(x^*)x^* = 0, \text{ for all } x\in R. \tag{2.8}$$

According to (2.3) we can obtain from (2.8) that

$$A_n(x)(x+x^*) = 0, \text{ for all } x\in R. \tag{2.9}$$

Replacing y by x in (2.5) we get

$$A_n(x)(x-x^*) = 0, \text{ for all } x\in R. \tag{2.10}$$

Combining the equations (2.9) and (2.10) we get

$$A_n(x)x = 0, \text{ for all } x\in R. \tag{2.11}$$

Since $A_n(x)\in Z(R)$, for all $x\in R$, we get

$$xA_n(x) = 0, \text{ for all } x\in R. \tag{2.12}$$

Polarizing (2.11) gives

$$A_n(x)y + B_n(x,y)x + A_n(y)x + B_n(x,y)y = 0 \text{ for all } x,y\in R, \tag{2.13}$$

where $B_n(x,y)$ denotes $f_n(xy+yx)-\sum_{i+j=n} f_i(x)d_j(y)+f_i(y)d_j(x)$. Putting $-x$ for x in (2.13) we get

$$A_n(x)y + B_n(x,y)x - A_n(y)x - B_n(x,y)y = 0 \text{ for all } x,y\in R. \tag{2.14}$$

Adding (2.13) and (2.14) we get, since R is 2-torsion free, that

$$A_n(x)y + B_n(x,y)x = 0 \text{ for all } x,y\in R. \tag{2.15}$$

Multiplying (2.15) by $A_n(x)$ from right and using (2.12) we get for all $x,y\in R$ that $A_n(x)yA_n(x)=0$. By the semiprimeness of R, we get $A_n(x)=0$ for all $x\in R$. This shows that F is a generalized Jordan higher derivation. In view of Lemma 2.2, F is a generalized higher derivation. This completes the proof of the theorem. $\square$

By Theorem 2.3, we immediately get the following corollary.

Corollary 2.4 ([7]). *Let R be a 2-torsion free semiprime $*-$ring and $F:R\to R$ an additive mapping associated with a derivation $d:R\to R$ such that $F(xx^*)=F(x)x^*+xd(x^*)$ holds for all $x\in R$. Then F is a generalized derivation.*

In the following theorem we get the same conclusion as in Theorem 2.3.

Theorem 2.5. *Let R be a 2-torsion free semiprime $*-$ring. Suppose there exists a family of additive mappings $F = (f_i)_{i\in\mathbb{N}_0}$ of R related with some higher derivation $D = (d_i)_{i\in\mathbb{N}_0}$ of R such that $f_0 = id_R$, and the relation $f_n(xy^*x) = \sum_{i+j+k=n} f_i(x)d_j(y^*)d_k(x)$ is fulfilled for each $n \in \mathbb{N}_0$ and for all $x, y \in R$. Then F is a generalized higher derivation.*

Proof. Our hypothesis is

$$f_n(xy^*x) = \sum_{i+j+k=n} f_i(x)d_j(y^*)d_k(x), \text{ for all } x, y \in R. \tag{2.16}$$

Putting $x = x + z$ in (2.16) and using (2.16) give for all $x, y, z \in R$

$$f_n(xy^*z + zy^*x) = \sum_{i+j+k=n} f_i(x)d_j(y^*)d_k(z) + f_i(z)d_j(y^*)d_k(x). \tag{2.17}$$

As in the proof of Theorem 2.3, for each fixed $n \in \mathbb{N}$ and every $x \in R$ we denote by $A_n(x)$ the elements of R defined by $f_n(x^2) - \sum_{i+j} f_i(x)d_j(x)$. Also we proceed by induction to show that $A_n(x) = 0$ for all $x \in R$ and for each positive integer n. If $n = 0$, it is easy to obtain $A_0(x) = 0$ for all $x \in R$. When $n = 1$, by [7, Theorem 2.1.] we know that $A_1(x) = 0$ for all $x \in R$. We now suppose in all the next steps that $A_m(x) = 0$ for all $x \in R$ and each positive integer $m < n$.

Replacing z by x^2 in (2.17) and using our assumption we get for all $x, y \in R$

$$\begin{aligned} f_n(xy^*x^2 + x^2y^*x) &= \sum_{i+j+k=n} f_i(x)d_j(y^*)d_k(x^2) + f_i(x^2)d_j(y^*)d_k(x) \\ &= \sum_{i+j+s+t=n} f_i(x)d_j(y^*)d_s(x)d_t(x) + f_n(x^2)y^*x \\ &\quad + \sum_{\substack{u+v+j+k=n \\ u+v\neq n}} f_u(x)d_v(x)d_j(y^*)d_k(x). \end{aligned}$$

Further, replacing $(x^*y + yx^*)$ for y in (2.16), we obtain for all $x, y \in R$

$$\begin{aligned} f_n(xy^*x^2 + x^2y^*x) &= \sum_{i+j+k=n} f_i(x)d_j(y^*x)d_k(x) + \sum_{i+j+k=n} f_i(x)d_j(xy^*)d_k(x) \\ &= \sum_{i+l+r+k=n} f_i(x)d_l(y^*)d_r(x)d_k(x) + \sum_{i+p=n} f_i(x)d_p(x)y^*x \\ &\quad + \sum_{\substack{i+p+q+k=n \\ i+p\neq n}} f_i(x)d_p(x)d_q(y^*)d_k(x). \end{aligned}$$

Comparing the last two equations and using our notation, we obtain

$$A_n(x)y^*x = 0, \text{ for all } x, y \in R. \tag{2.18}$$

By the surjectiveness of the involution the previous equation reduces to $A_n(x)yx = 0$ for all $y, x \in R$. Now, replacing y by $xyA_n(x) = 0$, we have $A_n(x)xyA_n(x)x = 0$ for all $x, y \in R$. By the semiprimeness of R, we get

$$A_n(x)x = 0, \text{ for all } x \in R. \tag{2.19}$$

On the other hand, multiplying $A_n(x)yx = 0$ by $A(x)$ from right and by x from left we get $xA_n(x)yxA_n(x) = 0$ for all $x, y \in R$. Again, by the semiprimeness of R we get

$$xA_n(x) = 0, \text{ for all } x \in R. \tag{2.20}$$

Linearizing (2.19) we get

$$A_n(x)y + B_n(x,y)x + A_n(y)x + B_n(x,y)y = 0 \text{ for all } x, y \in R, \tag{2.21}$$

where $B_n(x,y)$ denotes $f_n(xy+yx) - \sum_{i+j=n} f_i(x)d_j(y^{*^i}) - \sum_{i+j=n} f_i(y)d_j(x^{*^i})$. Putting $-x$ for x in (2.21) we get

$$A_n(x)y + B_n(x,y)x - A_n(y)x - B_n(x,y)y = 0 \text{ for all } x, y \in R. \tag{2.22}$$

Adding (2.21) and (2.22) we get, since R is 2-torsion free, that

$$A_n(x)y + B_n(x,y)x = 0 \text{ for all } x, y \in R.$$

Multiplying the last equation by $A_n(x)$ from right and using (2.20) we get $A_n(x)yA_n(x) = 0$ for all $x, y \in R$. By the semiprimeness of R, we get $A_n(x) = 0$ for all $x \in R$. In other words $F = (f_i)_{i \in \mathbb{N}_0}$ is a generalized Jordan higher derivation of R. In view of Lemma 2.2 we get the required result. □

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References

1. M. Ashraf, S. Ali, and C. Haetinger, *On derivations in rings and their applications*, Aligarh Bull. Math. **25** (2006), no. 2, 79–107.
2. M. Ashraf and A. Khan, *Generalized (σ, τ)–higher derivations in prime rings*, SpringerPlus **1** (2012), no. 1, 1–9.
3. ______, *On generalized Jordan triple (σ, τ)–higher derivations in prime rings*, ISRN Algebra **2014** (2014), Article ID 684792 8 pages.
4. M. Brešar, *On the distance of the composition of the two derivations to the generalized derivations*, Glasgow Math. J. **33** (1991), no. 1, 89–93.
5. W. Cortes and C. Haetinger, *On Jordan generalized higher derivations in rings*, Turk. J. Math. **29** (2005), 1–10.
6. J. M. Cusack, *Jordan derivations on rings*, Proc. Amer. Math. Soc. **53** (1975), no. 2, 321–324.
7. M. N. Daif and M. S. El-Sayiad, *On generalized derivations of semiprime rings with involution*, Internat. J. of Algebra **1** (2007), no. 12, 551–555.
8. O. H. Ezzat, *A note on Jordan triple higher ∗−derivations in semiprime rings*, ISRN algebra **2014** (2014), Article ID 365424, 5 pages.
9. ______, *Generalized Jordan triple higher ∗−derivaions on semiprime rings*, Gen. Math. Note **23** (2014), no. 2, 1–10.
10. M. Ferrero and C. Haetinger, *Higher derivations and a theorem by Herstein*, Quaest. Math. **25** (2002), 249–257.
11. C. Haetinger, M. Ashraf, and S. Ali, *On higher derivations : a survey*, Int. J. Math., Game Theory and Algebra **18** (2011), 359–379.
12. H. Hasse and F. K. Schmidt, *Noch eine begrüdung der theorie der höheren differential quotienten in einem algebraaischen funktionenkörper einer unbestimmeten*, J. Reine Angew. Math. **177** (1937), 215–237.

13. I. N. Herstein, *Jordan derivations of prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1104–1110.
14. J. Vukman and I. Kosi-Ulbl, *On centralizers of semiprime rings with involution*, Stud. Sci. Math. Hung. **43** (2006), no. 1, 61–67.
15. F. Wei and Z. Xiao, *Generalized Jordan derivations on semiprime rings and its applications in range inclusion problems*, Mediter. J. Math. **8** (2011), no. 3, 271–291.

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