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ON GENERALIZED RIGHT DERIVATIONS OF INCLINE ALGEBRAS

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ABSTRACT. In this paper, we introduce the concept of a generalized right derivation, which is a generalization of right derivation in incline algebras and give some properties of incline algebras. Also, the concept of D-ideal is introduced in an incline algebra with respect to a generalized right derivation.

1. Introduction

The notion of derivation on a ring has an important role for the characterization of rings. As a generalizations of derivations, α-derivations, generalized derivations on prime and semi-prime rings are studied by a lot of researchers. Z. Q. Cao, K. H. Kim and F. W. Roush [1] introduced the notion of incline algebras in their book and later it was developed by some authors [3, 4].

K. H. Kim [2] introduced the notion of right derivation in incline algebra. In this paper, we introduce the concept of a generalized right derivation, which is a generalization of right derivation in incline algebras and give some properties of incline algebras. Also, the concept of D-ideal is introduced in an incline algebra with respect to a generalized right derivation.

2. Preliminaries

An *incline algebra* is a set K with two binary operations denoted by "$+$" and "$*$" satisfying the following axioms:

(K1) $x + y = y + x$,
(K2) $x + (y + z) = (x + y) + z$,
(K3) $x * (y * z) = (x * y) * z$,
(K4) $x * (y + z) = (x * y) + (x * z)$,
(K5) $(y + z) * x = (y * x) + (z * x)$,
(K6) $x + x = x$,
(K7) $x + (x * y) = x$,
(K8) $y + (x * y) = y$, for all $x, y, z \in K$.

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For convenience, we pronounce "+" (resp. "$*$") as addition (resp. multiplication). Every distributive lattice is an incline algebra. An incline algebra is a distributive lattice if and only if $x * x = x$ for all $x \in K$. Note that $x \leq y \Leftrightarrow x + y = y$ for all $x, y \in K$. It is easy to see that " $\leq$ " is a partial order on K and that for any $x, y \in K$, the element $x + y$ is the least upper bound of $\{x, y\}$. We say that $\leq$ is induced by operation $+$.

In an incline algebra K, the following properties hold.

(K9) $x * y \leq x$ and $y * x \leq x$ for all $x, y \in K$,
(K10) $x \leq x + y$ and $y \leq x + y$, for all $x, y \in K$,
(K11) $y \leq z$ implies $x * y \leq x * z$ and $y * x \leq z * x$, for all $x, y, z \in K$,
(K12) If $x \leq y$ and $a \leq b$, then $x+a \leq y+b$, and $x*a \leq y*b$ for all $x, y, a, b \in K$.

Furthermore, an incline algebra K is said to be *commutative* if $x * y = y * x$ for all $x, y \in K$.

A *subincline* of an incline algebra K is a non-empty subset M of K which is closed under the addition and multiplication. A subincline M is called an *ideal* if $x \in M$ and $y \leq x$ then $y \in M$. An element "0" in an incline algebra K is a *zero element* if $x + 0 = x = 0 + x$ and $x * 0 = 0 = 0 * x$ for any $x \in K$. An non-zero element "1" is called a *multiplicative identity* if $x * 1 = 1 * x = x$ for any $x \in K$. A non-zero element $a \in K$ is said to be a *left* (resp. *right*) *zero divisor* if there exists a non-zero $b \in K$ such hat $a * b = 0$ (resp. $b * a = 0$). A zero divisor is an element of K which is both a left zero divisor and a right zero divisor. An incline algebra K with multiplicative identity 1 and zero element 0 is called an *integral incline* if it has no zero divisors. By a homomorphism of inclines, we mean a mapping f from an incline algebra K into an incline algebra L such that $f(x + y) = f(x) + f(y)$ and $f(x * y) = f(x) * f(y)$ for all $x, y \in K$.

Definition 2.1. Let K be an incline algebra. An element $a \in K$ is said to be *additively left cancellative* if for all $a, b \in K$, $a + b = a + c \Rightarrow b = c$. An element $a \in K$ is said to be *additively right cancellative* if for all $a, b \in K$, $b + a = c + a \Rightarrow b = c$. It is said to be *additively cancellative* if it is both left and right cancellative. If every element of K is additively left cancellative, it is called *additively left cancellative*. If every element of K is additively right cancellative, it is called *additively right cancellative*.

Definition 2.2. Let K be an incline algebra. By a *right derivation* of K, we mean a self map d of K satisfying the identities

$$d(x + y) = d(x) + d(y) \text{ and } d(x * y) = (d(x) * y) + (d(y) * x)$$

for all $x, y \in K$.

Proposition 2.3. *([2]) Let K be an incline algebra with a zero element and let d be a right derivation of K. Then $d(0) = 0$.*

Proposition 2.4. *([2]) Let K be a distributive lattice and let d be a right derivation of an incline algebra K. Then we have*

(1) $d(x) \leq x$ *for all* $x \in K$,

(2) $d(x * y) \leq d(x + y)$ *for all* $x, y \in K$,
(3) $d(x * y) \leq d(x)$ *and* $d(x * y) \leq d(y)$ *for all* $x, y \in K$.

3. Generalized right derivations of incline algebras

In what follows, let K denote an incline algebra with a zero-element unless otherwise specified.

Definition 3.1. Let K be an incline algebra. A function $D : K \to K$ is called a *generalized right derivation* on K if there exists a right derivation $d : K \to K$ such that

$$D(x + y) = D(x) + D(y) \text{ and } D(x * y) = (D(x) * y) + (d(y) * x)$$

for all $x, y \in K$.

Example 3.2. Let $K = \{0, a, b, 1\}$ be a set in which "+" and " $*$ " is defined by

$+$	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

$*$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

Then it is easy to check that $(K, +, *)$ is an incline algebra. Define a map $d : K \to K$ by

$$d(x) = \begin{cases} a & \text{if } x = a, b, 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Then we can see that d is a right derivation of X. Now, define a map $D : K \to K$ by

$$D(x) = \begin{cases} a & \text{if } x = a \\ b & \text{if } x = b, 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Then we can see that D is a generalized right derivation of K.

Proposition 3.3. *Let D be a generalized right derivation of an incline algebra K. Then we have $D(0) = 0$.*

Proof. Let D be a generalized right derivation of an incline algebra K. Then we have

$$D(0) = D(0 * 0) = D(0) * 0 + d(0) * 0 = 0 + 0 = 0.$$

□

Proposition 3.4. *Let K be a distributive lattice and let D be a generalized right derivation of an incline algebra K. Then we have*

(1) $d(x) \leq D(x) \leq x$ *for all* $x \in K$,
(2) $D(x * y) \leq D(x) + D(y)$ *for all* $x, y \in K$,
(3) *If* $x \leq y$, *then* $D(x * y) \leq y$,

(4) *If I is an ideal of K, then $D(I) \subseteq I$.*

Proof. (1) Let D be a generalized right derivation of K and let K be a distributive lattice. Then for $x \in K$, from (K10), (K12) and Proposition 2.4 (1),

$$D(x) = D(x * x) = D(x) * x + d(x) * x \geq d(x) * x \geq d(x) * d(x) = d(x).$$

Thus $d(x) \leq D(x)$. Also, since $D(x) = D(x * x) = D(x) * x + d(x) * x$, we get from (K8),

$$\begin{aligned} D(x) + x &= ((D(x) * x) + (d(x) * x)) + x \\ &= (D(x) * x) + (x + d(x) * x) \\ &= (D(x) * x) + x = x + (D(x) * x) \\ &= x. \end{aligned}$$

which implies $D(x) \leq x$ from (K9) for all $x \in K$.

(2) Let $x, y \in K$. Then from (K9), we have $D(x)*y \leq D(x)$ and $d(y)*x \leq d(y)$. Then by (K11) and (1), we have

$$D(x * y) = (D(x) * y) + (d(y) * x) \leq D(x) + d(y) \leq D(x) + D(y).$$

(3) Let $x \leq y$. Then by (K11) and (K9), we have $d(y) * x \leq d(y) * y \leq y$. Also, we obtain $D(x) * y \leq y$. Thus from (K12), we have

$$D(x * y) = (D(x) * y) + (d(y) * x) \leq y + y = y.$$

(4) Let I be an ideal of K and $x \in I$. Then by (1), $D(x) \leq x \in I$, and so $D(x) \in I$, which implies $D(I) \subseteq I$. □

Proposition 3.5. *Let K be an incline algebra and let D be a generalized right derivation of K. Then we have $D(x * y) \leq D(x) + d(y)$ for all $x, y \in K$.*

Proof. Let $x, y \in K$. By using (K9), we get $D(x) * y \leq D(x)$ and $d(y) * x \leq d(y)$. Thus we get

$$D(x * y) = (D(x) * y) + (d(y) * x) \leq D(x) + d(y).$$

□

Proposition 3.6. *Let D be a generalized right derivation of a distributive lattice K. If $d(D(x)) = D(x)$, then $D(x * D(x)) = D(x)$ for all $x \in K$.*

Proof. Let K be an incline algebra and $x \in K$. Then from (K7),

$$\begin{aligned} D(x * D(x)) &= (D(x) * D(x)) + (d(D(x)) * x) \\ &= D(x) + (D(x) * x) \\ &= D(x). \end{aligned}$$

□

Proposition 3.7. *Let K be an incline algebra and let D be a generalized right derivation of K. Then for all $x, y \in K$, $D(x * y) \leq D(x)$ and $D(x * y) \leq D(y)$.*

Proof. Let $x, y \in K$. Then by using (K7), we obtain

$$D(x) = D(x + x * y) = D(x) + D(x * y).$$

Hence we get $D(x * y) \leq D(x)$. Also, $D(y) = D(y + (x * y)) = D(y) + D(x * y)$, and so $D(x * y) \leq D(y)$. □

Definition 3.8. Let K be an incline algebra. A mapping $f : K \to K$ is *isotone* if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in K$.

Proposition 3.9. *Let D be a generalized right derivation of an incline algebra K. Then D is isotone.*

Proof. Let $x, y \in K$ be such that $x \leq y$. Then $x + y = y$. Hence we have $D(y) = D(x + y) = D(x) + D(y)$, which implies $D(x) \leq D(y)$. This completes the proof. □

Proposition 3.10. *A sum of two generalized right derivations associated with right derivations d_1 and d_2 of K is again a generalized right derivation of K.*

Proof. Let D_1 and D_2 be two generalized right derivations of K respectively. Then we have for all $a, b \in K$,

$$\begin{aligned}
(D_1 + D_2)(a * b) &= D_1(a * b) + D_2(a * b) \\
&= D_1(a) * b + d_1(b) * a + D_2(a) * b + d_2(b) * a \\
&= D_1(a) * b + D_2(a) * b + d_1(b) * a + d_2(b) * a \\
&= (D_1 + D_2)(a) * b + (d_1 + d_2)(b) * a.
\end{aligned}$$

Clearly, $(D_1 + D_2)(a + b) = (D_1 + D_2)(a) + (D_1 + D_2)(b)$ for all $a, b \in K$. This completes the proof. □

Theorem 3.11. *Let K be a commutative incline algebra and let D_1 and D_2 be a generalized right derivations of K associated with right derivations d_1 and d_2 of K. Define $D_1D_2(x) = D_1(D_2(x))$ for all $x \in K$. If $D_1D_2 = 0$, then D_2D_1 is a generalized right derivation of K.*

Proof. Let K be a commutative incline algebra and $D_1D_2 = 0$. Then we have for all $x, y \in K$,

$$\begin{aligned}
0 = D_1D_2(x * y) &= D_1(D_2(x) * y + d_2(y) * x) \\
&= D_1(D_2(x) * y + x * d_2(y)) \\
&= D_1(D_2(x)) * y + d_1(y) * D_2(x) + D_1(x) * d_2(y) + d_1(d_2(y)) * x.
\end{aligned}$$

By Proposition 3.4 (1), we have $d_1d_2 \leq D_1D_2 = 0$, and so $d_1d_2 = 0$. Hence we have $D_2(x) * d_1(y) + D_1(x) * d_2(y) = 0$. Also, we get

$$\begin{aligned}
D_2D_1(x * y) &= D_2(D_1(x) * y + d_1(y) * x) \\
&= D_2(D_1(x) * y + x * d_1(y)) \\
&= D_2D_1(x) * y + d_2(y) * D_1(x) + D_2(x) * x + d_1(y) * d_2d_1(y) \\
&= D_2D_1(x) * y + d_2d_1(y) * x.
\end{aligned}$$

Finally, for all $x, y \in K$, we get

$$D_2D_1(x+y) = D_2(D_1(x) + D_1(y)) = D_2D_1(x) + D_2D_1(y).$$

This implies that D_2D_1 is a generalized right derivation of a commutative incline algebra K. □

Theorem 3.12. *Let M be a non-zero ideal of an integral incline K. If d is a non-zero right derivation of K, D is a non-zero generalized right derivation on M.*

Proof. Suppose that D is a non-zero generalized right derivation of K but D is a zero generalized right derivation of M. Let $x \in M$. Then $D(x) = 0$. Now, let $y \in K$. By (K9), $x * y \leq x$ and since M is an ideal of an integral incline K, we get $x * y \in M$, which means $D(x * y) = 0$. Hence we get

$$0 = D(x * y) = (D(x) * y) + (d(y) * x) = d(y) * x.$$

We know by our assumption that K has no zero-divisors. Hence, we have $x = 0$ for all $x \in M$ or $d(y) = 0$ for all $y \in K$. Since M is a nonzero ideal of K, we have $d(y) = 0$ for all $y \in K$. Since $M \neq 0$, we have $d(y) = 0$ for all $y \in K$. This contradicts $d \neq 0$ on K. Hence, D is a non-zero generalized right derivation on M. □

Theorem 3.13. *Let K be a commutative integral incline and let D be a generalized right derivation of K. Define $D^2(x) = D(D(x))$ for all $x \in K$. If $D^2 = 0$, then D is zero.*

Proof. Let $x, y \in K$. Then

$$\begin{aligned} 0 &= D^2(x * y) = D(D(x * y)) \\ &= D(D(x) * y + d(y) * x) = D(D(x) * y + x * d(y)) \\ &= D^2(x) * y) + d(y) * D(x) + D(x) * d(y) + d^2(y) * x \\ &= D^2(x) * y) + D(x) * d(y) + D(x) * d(y) + d^2(y) * x. \end{aligned}$$

By Proposition 3.4 (1), we have $d^2 \leq D^2 = 0$, and so $d^2 = 0$. Hence we have

$$0 = D(x) * d(y) + D(x) * d(y) = D(x) * d(y).$$

Since K has no zero divisors, we get $D(x) = 0$ for all $x \in K$ or $d(y) = 0$ for all $y \in K$. Thus, in two cases, we have $D = 0$ □

Theorem 3.14. *If D is a nonzero generalized right derivation and $M \neq 0$ is an ideal of a commutative integral incline K, then $D^2(M) \neq 0$.*

Proof. Let $D^2(M) = 0$. Then for $x, y \in M$, we have

$$\begin{aligned} 0 &= D^2(x * y) = D(D(x * y)) \\ &= D(D(x) * y + d(y) * x) = D(D(x) * y + x * d(y)) \\ &= D^2(x) * y + d(y) * D(x) + D(x) * d(y) + d^2(y) * x. \end{aligned}$$

By Proposition 3.4 (1), we have $d^2 \leq D^2 = 0$, and so $d^2 = 0$. Hence we have

$$0 = D(x) * d(y) + D(x) * d(y) = D(x) * d(y).$$

Since K has no zero divisors, we get $D(x) = 0$ for all $x \in K$ or $d(y) = 0$ for all $y \in K$. Thus, in two cases, we have $D = 0$. Hence $D(M) = 0$, which implies that D is a zero generalized right derivation of M. This contradicts $D \neq 0$. Hence D^2 must be different than zero on M. $\square$

Theorem 3.15. *Let K be an integral incline and let D be a generalized right derivation of K associated with a nonzero right derivation d of K. Then, $a * D(x) = 0$ implies $a = 0$ or $d = 0$ for all $x \in K$.*

Proof. Let $a * D(x) = 0$ for all $x \in K$. If we replace x by $x * y$ for $y \in K$, we get

$$\begin{aligned} 0 = a * D(x) = a * D(x * y) &= [(D(x) * y) + (d(y) * x)] \\ &= (a * (D(x) * y)) + (a * (d(y) * x)) \\ &= a * (d(y) * x). \end{aligned}$$

In this equation, by taking $x = 1$, we get $a * d(y) = 0$. Since K is an integral incline, we have $a = 0$ or $d(y) = 0$ for all $y \in K$. Thus we have $a = 0$ or $d = 0$. $\square$

Theorem 3.16. *Let K be a commutative integral incline and let D be a generalized right derivation of K associated with a nonzero right derivation d of K. Then $D(x) * a = 0$ implies that $a = 0$ or $d = 0$ for all $x \in K$.*

Proof. Let $D(x) * a = 0$ for all $x \in K$. If we replace x by $x * y$ for $y \in K$, we obtain

$$\begin{aligned} 0 = D(x) * a &= [(D(x) * y) + (d(y) * x)] * a \\ &= ((D(x) * y) * a) + ((d(y) * x) * a) \\ &= ((D(x) * a) * y) + ((x * (d(y) * a)) \\ &= x * (d(y) * a). \end{aligned}$$

In this equation, by taking $x = 1$, we get $d(y) * a = 0$. Since K is an integral incline, we have $a = 0$ or $d(y) = 0$ for all $y \in K$. Thus we have $a = 0$ or $d = 0$. $\square$

Theorem 3.17. *Let D be a nonzero generalized right derivation of an integral incline K associated with a nonzero right derivation d of K. If M is a non-zero ideal of integral incline K and $a \in K$ such that $a * D(M) = 0$, then $a = 0$.*

Proof. By Theorem 3.12, we know that there is an element m in M such that $D(m) \neq 0$. Let M be a nonzero ideal of K and $a * D(M) = 0$ for $a \in K$. Then for $m, n \in M$, we have

$$\begin{aligned} 0 &= a * D(m * n) \\ &= a * ((D(m) * n + (d(n) * m)) \\ &= a * (D(m) * n)) + a * (d(n) * m) \\ &= a * (d(n) * m). \end{aligned}$$

Since K is an integral incline, d is a non-zero right derivation on M and M is a non-zero ideal, we have $a = 0$. $\square$

Let D be a generalized right derivation of K. Define a set $Fix_D(K)$ by

$$Fix_D(K) := \{x \in K \mid D(x) = x\}$$

for all $x \in K$.

Theorem 3.18. *Let K be an incline algebra and let D be an isotone generalized right derivation of K. If K is additively right cancellative, then $Fix_D(K)$ is an ideal of K.*

Proof. Let $x, y \in Fix_D(K)$. Then $D(x) = x$ and $D(y) = y$. Hence $D(x + y) = D(x) + D(y) = x + y$, which implies $x + y \in Fix_D(K)$. Also, we get

$$D(x * y) = (D(x) * y) + (d(y) * x) \geq D(x) * y = x * y$$

and from Proposition 3.4 (1), $D(x * y) \leq x * y$. Hence $D(x * y) = x * y$, which implies $x * y \in Fix_D(K)$. Now, let $x \in K$ and $y \in Fix_D(K)$ such that $x \leq y$. Then $D(y) = y$ and $y = x + y$. Since D is isotone, we get $D(x) \leq D(y)$, i.e., $D(x) + D(y) = D(y) = y = x + y = x + D(y)$. Hence $D(x)$ must be equal to x, which means $x \in Fix_D(K)$. □

Theorem 3.19. *Let K be an incline algebra and let D_1, D_2 be two isotone generalized right derivations on K. If $D(x) \in Fix_D(K)$, then $D_1 = D_2$ if and only if $Fix_{D_1}(K) = Fix_{D_2}(K)$.*

Proof. Let $D_1 = D_2$. Then $Fix_{D_1}(K) = Fix_{D_2}(K)$. Conversely, let $Fix_{D_1}(K) = Fix_{D_2}(K)$ and $D(x) \in Fix_D(K)$ for $x \in K$. Then $D_1(x) \in Fix_{D_1}(K) = Fix_{D_2}(K)$, and so $D_2(D_1(x)) = D_1(x)$. Also, $D_2(x) \in Fix_{D_2}(K) = Fix_{D_1}(K)$, and so $D_1(D_2(x)) = D_2(x)$. Since $D_1(x) \leq x$, we have $D_2(D_1(x)) \leq D_2(x)$, and so $D_2(D_1(x)) \leq D_2(x) = D_1(D_2(x))$. Symmetrically, we have $D_1(D_2(x)) \leq D_2(D_1(x))$. Hence $D_1D_2 = D_2D_1$. It follows that $D_2(x) = D_1(D_2(x)) = D_2(D_1(x)) = D_1(x)$. □

Definition 3.20. Let K be an incline algebra and let D be a generalized right derivation of K.

(1) D is called a *monomorphic generalized right derivation* if D is one-to- one.
(2) D is called an *epic generalized right derivation* if D is onto.

Theorem 3.21. *Let D be a generalized right derivation of K and $D(x) \in Fix_D(K)$. Then the following conditions are equivalent:*

(1) *$D(x) = x$ for all $x \in K$,*
(2) *D is a monomorphic generalized right derivation of K,*
(3) *D is an epic generalized right derivation of K.*

Proof. (1) ⇒(2) is clear.
(2) ⇒(1) Let D be a monomorphic generalized right derivation and $x \in K$. Since $D(x) \in Fix_D(K)$, we have $D(D(x)) = D(x)$. Since D is monomorphic, $D(x) = x$ for all $x \in K$.
(1) ⇒(3) is trivial.
(3) ⇒(1) Let D be an epic generalized right derivation and $x \in K$. Then there exists $y \in K$ such that $D(y) = x$. Since $D(x) \in Fix_D(K)$, we have $D(x) = D(D(y)) = D^2(y) = D(y) = x$. □

Definition 3.22. Let K be an incline algebra and let D be a non-trivial generalized right derivation of K. An ideal I of K is called a *D-ideal* if $D(I) = I$.

Since $D(0) = 0$, it can be easily observed that the zero ideal $\{0\}$ is a d-ideal of K. If D is onto, then $D(K) = K$, which implies that K is a D-ideal of K.

Example 3.23. In Example 3.2, let $I = \{0, a\}$. Then I is an ideal of K. It can be verified that $D(I) = I$. Therefore, I is an D-ideal of K.

Lemma 3.24. *Let D be a generalized right derivation of K and let I, J be any two D-ideals of K. Then $I \subseteq J$ implies $D(I) \subseteq D(J)$.*

Proof. Let $I \subseteq J$ and $x \in D(I)$. Then we have $x = D(y)$ for some $y \in I \subseteq J$. Hence we get $x = D(y) \in D(J)$. Therefore, $D(I) \subseteq D(J)$. □

Proposition 3.25. *Let K be an incline algebra. Then, a sum of any two D-ideals is also a D-ideal of K.*

Proof. Let I and J be D-ideals of K. Then $I + J = D(I) + D(J) = D(I + J)$. Hence $I + J$ is a D-ideal of K. □

Let D be a generalized right derivation of K. Define a set $KerD$ by

$$KerD := \{x \in K \mid D(x) = 0\}$$

for all $x \in K$.

Proposition 3.26. *Let D be a generalized right derivation of an incline algebra K. Then $KerD$ is a subincline of K.*

Proof. Let $x, y \in KerD$. Then $D(x) = 0$ and $D(y) = 0$. Since $d(y) \leq D(y) = 0$ from Proposition 3.4 (1), we get $d(y) = 0$. Thus we have

$$\begin{aligned} D(x * y) &= (D(x) * y) + (d(y) * x) \\ &= (0 * y) + (0 * x) \\ &= 0 + 0 = 0, \end{aligned}$$

which implies $x * y \in KerD$. Also, we obtain

$$\begin{aligned} D(x + y) &= D(x) + D(y) \\ &= 0 + 0 = 0. \end{aligned}$$

Therefore, $x * y, x + y \in KerD$. This completes the proof. □

Proposition 3.27. *Let D be a generalized right derivation of an integral incline algebra K. Then $KerD$ is an ideal of K.*

Proof. By Proposition 3.26, $KerD$ is a subincline of K. Now let $x \in K$ and $y \in KerD$ such that $x \leq y$. Then $D(y) = 0$ and $x + y = y$. Hence

$$0 = D(y) = D(x + y) = D(x) + D(y) = D(x) + 0 = D(x).$$

Hence $x \in KerD$. □

Let D be a generalized right derivation of K associated with a right derivation d of K. Define a set $D(K)$ by

$$D(K) := \{x \in K \mid D(x) = d(x)\}$$

for all $x \in K$.

Proposition 3.28. *Let K be an additively right cancellative incline algebra and let D be a generalized right derivation of K. If D is isotone, then $D(K)$ is an ideal of K.*

Proof. Let $x, y \in D(K)$. Then we have $D(x) = d(x)$ and $D(y) = d(y)$, and so

$$\begin{aligned} D(x * y) &= D(x) * y + d(y) * x \\ &= d(x) * y + d(y) * x = d(x * y). \end{aligned}$$

Now

$$D(x + y) = D(x) + D(y) = d(x) + d(y) = d(x + y),$$

which implies $x + y, x * y \in D(K)$. Now, let $x \in K$ and $y \in D(K)$ such that $x \leq y$. Then $D(y) = d(y)$ and $y = x + y$. Since D is isotone, we get $D(x) \leq D(y)$, i.e., $D(x) + D(y) = D(y) = d(y) = d(x) + d(y) = d(x) + D(y)$. Hence $D(x)$ must be equal to $d(x)$, which means $x \in D(K)$. This completes the proof. □

Proposition 3.29. *Let K be an incline algebra and let D be a generalized right derivation of K. If $x \in D(K)$ and $y \in K$, then $x * y \in D(K)$.*

Proof. Let $x \in D(K)$ and $y \in K$. Then $D(x) = d(x)$. Hence we have

$$D(x * y) = D(x) * y + d(y) * x = d(x) * y + d(y) * x = d(x * y),$$

which implies $x * y \in D(K)$. □

Definition 3.30. A subincline I of an incline algebra K is called a *k-ideal* if $x + y \in I$ and $y \in I$, then $x \in I$.

Example 3.31. In Example 3.2, $I = \{0, a, b\}$ is an k-ideal of K.

Theorem 3.32. *Let K be a commutative incline algebra and additively right cancellative. If D is a generalized right derivation of K, then $D(K)$ is a k-ideal of K.*

Proof. By part of proof in Proposition 3.24, $D(K)$ is a subincline of K. Let $x+y, y \in D(K)$. Then $D(y) = d(y)$ and $D(x+y) = d(x+y)$. Hence $D(x)+D(y) = D(x + y) = d(x + y) = d(x) + d(y) = d(x) + D(y)$, which implies $D(x) = d(x)$, i.e., $x \in D(K)$. Hence $D(K)$ is a k-ideal of K. □

Proposition 3.33. *Let K be an incline algebra and let D be a generalized right derivation of K. Then $KerD$ is a k-ideal of K.*

Proof. From Proposition 3.26, $KerD$ is a subincline of K. Let $x + y \in K$ and $y \in KerD$. Then we have $D(x + y) = 0$ and $D(y) = 0$, and so

$$0 = D(x + y) = D(x) + D(y) = D(x) + 0 = D(x).$$

This implies $x \in KerD$. □

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