Gulf Journal of Mathematics
Vol 3, Issue 1 (2015) 18-28

SLANT SUBMERSIONS FROM LORENTZIAN ALMOST PARACONTACT MANIFOLDS

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ABSTRACT. In this paper, we introduce slant submersions from Lorentzian almost paracontact manifolds onto Riemannian manifolds. We give examples and investigate the geometry of foliations which are arisen from the definition of a Lorentzian submersion. We also find necessary and sufficient conditions for a slant submersion to be totally geodesic.

1. INTRODUCTION

Let π be a C^∞-submersion from a semi-Riemannian manifold(M, g) onto a semi-Riemannian manifold (B, g'). Then according to the conditions on the map $\pi : (M, g) \to (B, g')$ we have the following submersions:
semi-Riemannian submersion and Lorentzian submersion [7], Riemannian submersion ([8],[18]), slant submersion ([9],[19],[22]), almost Hermitian submersion [24],almost contact submersion[6], paracontact submersion [10], paracontact paracomplex submersion [11], (para) quaternionic submersion ([15],[5]), anti-invariant submersion [23], H-semi-invariant submersion [20], etc. As we know, Riemannian submersions are related with physics and have their applications in the Yang-Mills theory ([4],[25]), Kaluza-Klein theory ([3],[12]), supergravity and superstring theories ([13],[14]), etc. On the other hand, the study of Lorentzian almost paracontact manifolds was initiated by Matsumoto in 1989 [16]. Slant submanifolds of Lorentzian almost paracontact manifolds were studied in [17].

In this paper, we define slant submersions of Lorentzian almost paracontact manifolds. The paper is organized as follows: In section 2 we recall some notions needed for this paper. In section 3 we give the definition of slant submersions and provide examples. We also investigate the geometry of leaves of the distributions. Finally we give necessary and sufficient conditions for such submersions to be totally geodesic.

Date: Received: Sep 19, 2014; Accepted: Dec 1, 2014.
2010 *Mathematics Subject Classification.* Primary 53C15; Secondary 53C40.
Key words and phrases. Lorentzian almost paracontact manifold, Lorentzian submersion, slant submersion.

2. Preliminaries

In this section, we are going to recall main definitions and properties of Lorentzian almost paracontact manifolds and Lorentzian submersions.

Let M be an $m-$dimensional differentiable manifold equipped with a triple (φ, ξ, η), where φ is a (1,1) tensor field, ξ a vector field, η is a 1-form on M such that

$$\eta(\xi) = -1, \tag{2.1}$$
$$\varphi^2 = I + \eta \otimes \xi, \tag{2.2}$$

where I denotes the identity map of T_pM and $\otimes$ is the tensor product. The equations (2.1) and (2.2) imply that

$$\eta \circ \varphi = 0,\ \varphi\xi = 0,\ rank(\varphi) = m - 1. \tag{2.3}$$

Then M admits a Lorentzian metric g, such that, for all $X, Y \in \chi(M)$,

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.4}$$

and M is said to admit a Lorentzian almost paracontact structure (φ, ξ, η, g). Then we get

$$g(X, \xi) = \eta(X), \tag{2.5}$$
$$\Phi(X, Y) = g(X, \varphi Y) = g(\varphi X, Y) = \Phi(Y, X), \tag{2.6}$$

where ∇ is the Riemannian connection with respect to g. It is clear that Lorentzian metric g makes ξ a timelike unit vector field, i.e, $g(\xi, \xi) = -1$. The manifold M equipped with a Lorentzian almost paracontact structure (φ, ξ, η, g) is called a Lorentzian almost paracontact manifold.
A Lorentzian almost paracontact manifold endowed with the structure (φ, ξ, η, g) is called a Lorentzian paraSasakian manifold [16] if

$$(\nabla_X \varphi)Y = \eta(Y)X + g(X, Y)\xi + 2\eta(X)\eta(Y)\xi. \tag{2.7}$$

A Lorentzian almost paracontact manifold is called a Lorentzian paracosymplectic manifold [21] if

$$\nabla\varphi = 0. \tag{2.8}$$

Let M be a Lorentzian manifold and B a Riemannian manifold. A Lorentzian submersion is defined to be a map $\pi : M \to B$ which is onto and satisfies the following three axioms (S1), (S2), and (S3).
(S1) π_{*p} is onto for all $p \in M$;
(S2) The fibres $\pi^{-1}(b)$ are semi-Riemannian submanifolds of M for each $b \in B$.
(S3) π_* preserves scalar products of horizontal vectors([1]).

The vectors tangent to fibres are called vertical and those normal to fibres are called horizontal. We denote by $\mathcal{V}$ the vertical distribution, by $\mathcal{H}$ the horizontal distribution and by v and h the vertical and horizontal projection. A horizontal vector field X on M is said to be basic if X is $\pi-$related to a vector field X' on B. It is clear that every vector field X' on B has a unique horizontal lift X to M

and X is basic.
The fundamental equations of a submersion have been developed by O'Neill[18]. Though O'Neill is considering Rimannian submersions, he notes that the basic equations are the same in the semi-Riemannian case.

We recall that the sections of $\mathcal{V}$, respectively $\mathcal{H}$, are called the vertical vector fields, respectively horizontal vector fields. A Lorentzian submersion $\pi : M \to B$ determines two $(1,2)$ tensor fields T and A on M, by the formulas:

$$T(E,F) = T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF \tag{2.9}$$

and

$$A(E,F) = A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF \tag{2.10}$$

for any $E, F \in \Gamma(TM)$, where v and h are the vertical and horizontal projections ([7]). From (2.9) and (2.10), one can obtain

$$\nabla_U X = T_U X + h(\nabla_U X); \tag{2.11}$$

$$\nabla_X U = v(\nabla_X U) + A_X U; \tag{2.12}$$

$$\nabla_X Y = A_X Y + h(\nabla_X Y), \tag{2.13}$$

for any $X, Y \in \Gamma(\mathcal{H})$, $U \in \Gamma(\mathcal{V})$. Moreover, if X is basic then $h(\nabla_U X) = h(\nabla_X U) = A_X U$.

We note that for $U, V \in \Gamma(\mathcal{V})$, $T_U V$ coincides with the second fundamental form of the immersion of the fibre submanifolds and for $X, Y \in \Gamma(\mathcal{H})$, $A_X Y = \frac{1}{2} v[X,Y]$ reflecting the complete integrability of the horizontal distribution $\mathcal{H}$. It is known that A is alternating on the horizontal distribution: $A_X Y = -A_Y X$, for $X, Y \in \Gamma(\mathcal{H})$ and T is symmetric on the vertical distribution: $T_U V = T_V U$, for $U, V \in \Gamma(\mathcal{V})$.

We now recall the following result which will be useful for later.

Lemma 2.1. *(see [7]). If $\pi : M \to B$ is a Lorentzian submersion and X, Y basic vector fields on M, $\pi-$related to X' and Y' on B, then we have the following properties*

1. *$h[X,Y]$ is a basic vector field and $\pi_* h[X,Y] = [X', Y'] \circ \pi$;*
2. *$h(\nabla_X Y)$ is a basic vector field $\pi-$related to $(\nabla'_{X'} Y')$, where ∇ and ∇' are the Levi-Civita connection on M and B;*
3. *$[E, U] \in \Gamma(ker\pi_*)$, for any $U \in \Gamma(ker\pi_*)$ and for any basic vector field E.*

Let (M, g) and (N, g') be (semi)-Riemannian manifolds and $\pi : M \to N$ is a smooth map. Then the second fundamental form of π is given by

$$(\nabla\pi_*)(X,Y) = \nabla^{\pi}_X \pi_* Y - \pi_*(\nabla_X Y) \tag{2.14}$$

for $X, Y \in \Gamma(TM)$, where we denote conveniently by ∇ the Levi-Civita connections of the metrics g and g'. Recall that π is said to be *harmonic* if $trace(\nabla\pi_*) = 0$ and π is called a *totally geodesic* map if $(\nabla\pi_*)(X,Y) = 0$ for $X, Y \in \Gamma(TM)$[2]. It is known that the second fundamental form is symmetric.

3. Slant Submersions

In this section, we define slant submersions from a Lorentzian almost paracontact manifold onto a Riemannian manifold by using the definition of a slant distribution given in [17]. We give examples, investigate the geometry of leaves of distributions. We also obtain a necessary and sufficient condition for such submersions to be totally geodesic map.

Definition 3.1. Let π be a Lorentzian submersion from a Lorentzian almost paracontact manifold $(M_1, g_1, \varphi, \xi, \eta)$ onto a Riemannian manifold (M_2, g_2). If for any non-zero vector $X \in ker\pi_* - sp\{\xi_p\}$; $p \in M_1$, the angle $\theta(X)$ between φX and the space $(ker\pi_* - sp\{\xi_p\})$ is a constant, i.e. it is independent of the choice of the point $p \in M_1$ and choice of the tangent vector X in $(ker\pi_* - sp\{\xi_p\})$, then we say that π is a slant submersion. In this case, the angle θ is called the slant angle of the slant submersion.

It is known that the distribution $(ker\pi_*)$ is integrable for a Lorentzian submersion. In fact, its leaves are $\pi^{-1}(p)$, $p \in M_1$, i.e., fibers. Thus it follows from above definition that the fibers of a slant submersion are slant submanifolds of M_1, for slant submanifolds, (see: [17]).

We note that the characteristic vector field ξ is a vertical vector field.

The Lorentzian almost paracontact structure (φ, g, ξ, η) is defined on R^5 in the following way:

$$\begin{aligned}
\varphi(\frac{\partial}{\partial x_1}) &= \frac{\partial}{\partial x_2},\ \varphi(\frac{\partial}{\partial x_2}) = \frac{\partial}{\partial x_1},\ \varphi(\frac{\partial}{\partial y_1}) = \frac{\partial}{\partial y_2},\ \varphi(\frac{\partial}{\partial y_2}) = \frac{\partial}{\partial y_1}, \\
\varphi(\xi) &= 0,\ \xi = -\frac{\partial}{\partial z},\ \eta = dz.
\end{aligned}$$

If $Z = a_i(\frac{\partial}{\partial x_i}) + b_i(\frac{\partial}{\partial y_i}) + v(\frac{\partial}{\partial z}) \in T(R^5)$, then we have

$$g(Z, Z) = \sum_{i=1}^{2} a_i^2 + \sum_{i=1}^{2} b_i^2 - v^2.$$

From this definition, it follows that

$$g(\varphi Z, \varphi Z) = g(Z, Z) + \eta^2(Z),\ \varphi\xi = 0,\ \eta(\xi) = -1$$

for an arbitrary vector field Z. Thus $(R^5, \varphi, g, \xi, \eta)$ becomes a Lorentzian almost paracontact manifold, where g and $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z}\}$ denote Lorentzian metric and standard basis of $T(R^5)$, respectively.

We first give some examples of slant submersions.

Example 3.2. Every anti-invariant Lorentzian submersion from a Lorentzian almost paracontact manifold onto a Riemannian manifold is a slant submersion with $\theta = 90$.

Example 3.3. Consider the following submersion given by

$$\begin{array}{rcl} \pi : R^5 & \to & R^2 \\ (x_1, x_2, y_1, y_2, z) & \to & (\frac{x_1 - y_1}{\sqrt{2}}, y_2). \end{array}$$

Then π is a slant submersion with slant angle $\theta = 45$.

Example 3.4. Define a map $\pi : R^5 \to R^2$ by

$$\pi(x_1, x_2, y_1, y_2, z) = (x_1 \sin\alpha - y_1 \cos\alpha, y_2),$$

where $0 < \alpha < 90$. Then the map π is a slant submersion with the slant angle $\theta = \alpha$.

Example 3.5. Define a map $\pi : R^5 \to R^2$ by

$$\pi(x_1, x_2, y_1, y_2, z) = (x_1 \cos\alpha - y_1 \sin\alpha, x_2 \sin\beta - y_2 \cos\beta).$$

Then the map π is a slant submersion with the slant angle θ with
$\cos\theta = |\sin(\alpha + \beta)|$.

Example 3.6. Define a map $\pi : R^5 \to R^2$ by

$$\pi(x_1, x_2, y_1, y_2, z) = (x_1 \cos\alpha + y_1 \sin\alpha, -x_2 \cos\alpha + y_2 \sin\alpha),$$

where $0 < \alpha < 90$. Then the map π is a slant submersion with the slant angle $\cos\theta = |\cos 2\alpha|$

Let π be a Lorentzian submersion from a Lorentzian almost paracontact manifold M_1 with the structure $(g_1, \varphi, \xi, \eta)$ onto a Riemannian manifold (M_2, g_2). Then for $X \in \Gamma(ker\pi_*)$, we write

$$\varphi X = \phi X + \omega X, \tag{3.1}$$

where ϕX and ωX are vertical and horizontal parts of φX. From (2.6) and (3.1), one can easily see that

$$g_1(X, \phi Y) = g_1(\phi X, Y), \tag{3.2}$$

for any $X, Y \in \Gamma(ker\pi_*)$.

Also for $Z \in \Gamma((ker\pi_*)^\perp)$, we have

$$\varphi Z = BZ + CZ, \tag{3.3}$$

where BZ and CZ are vertical and horizontal component of φZ. From (2.6) and (3.3), one can easily see that

$$g_1(Z_1, CZ_2) = g_1(CZ_1, Z_2), \tag{3.4}$$

for any $Z_1, Z_2 \in \Gamma((ker\pi_*)^\perp)$.

Span$\{\xi\}$ defines the timelike unit vector field distribution. If $X \in ker\pi_*$ is a spacelike vector field, which is orthogonal to ξ, then

$$g(\varphi X, \varphi X) = g(X, X) \geq 0,$$

so φX is also spacelike, the same is valid for ϕX. For spacelike vector fields the Cauchy-Schawrz inequality, $g(X,Y) \leq |X||Y|$, is verified. Therefore the Wirtinger angle, θ, is given by:

$$\frac{g(\varphi X, \phi X)}{|\varphi X||\phi X|} = \cos\theta.$$

We define the covariant derivatives of ϕ and ω as follows

$$(\nabla_X \phi)Y = \hat{\nabla}_X \phi Y - \phi \hat{\nabla}_X Y \tag{3.5}$$

and

$$(\nabla_X \omega)Y = h\nabla_X \omega Y - \omega \hat{\nabla}_X Y \tag{3.6}$$

for $X, Y \in \Gamma(ker\pi_*)$, where $\hat{\nabla}_X Y = v\nabla_X Y$. Then we easily have

Lemma 3.7. *Let $(M_1, g_1, \varphi, \xi, \eta)$ be a lorentzian almost paracontact manifold and (M_2, g_2) a Riemannian manifold. Let $\pi : M_1 \to M_2$ be a slant submersion. Then we get*

$$\begin{aligned} \hat{\nabla}_X \phi Y + T_X \omega Y &= \phi \hat{\nabla}_X Y + BT_X Y \\ T_X \phi Y + h\nabla_X \omega Y &= \omega \hat{\nabla}_X Y + CT_X Y \end{aligned}$$

for any $X, Y \in \Gamma(ker\pi_*)$.

Let π be a slant submersion from a Lorentzian almost paracontact manifold $(M_1, g_1, \varphi, \xi, \eta)$ onto a Riemannian manifold (M_2, g_2) with the slant angle $\theta \in (0, 90)$, then we say that ω is parallel with respect to the Levi-Civita connection ∇ on $(ker\pi_*)$ if its covariant derivative with respect to ∇ vanishes, i.e., we have

$$(\nabla_X \omega)Y = h\nabla_X \omega Y - \omega \hat{\nabla}_X Y = 0 \tag{3.7}$$

for $X, Y \in \Gamma(ker\pi_*)$.

Invariant and anti-invariant submanifolds are particular classes of slant submanifolds with slant angles $\theta = 0$ and $\theta = 90$, respectively. A slant submanifold which is neither invariant nor anti-invariant submanifold is called a proper slant submanifold([17]).

Theorem 3.8. *Let π be a Lorentzian submersion from a Lorentzian almost paracontact manifold $(M_1, g_1, \varphi, \xi, \eta)$ onto a Riemannian manifold (M_2, g_2). Then π is a proper slant submersion if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$\phi^2 X = \lambda(X + \eta(X)\xi)$$

for $X \in \Gamma(ker\pi_*)$. If π is a proper slant submersion, then $\lambda = \cos^2\theta$.

Proof. For any nonzero $X \in \Gamma(ker\pi_*)$, we can write

$$\cos\theta(X) = \frac{\|\phi X\|}{\|\varphi X\|}, \tag{3.8}$$

where $\theta(X)$ is the slant angle. By using (3.2), (3.8) and (2.2) we get

$$\begin{aligned} g_1(\phi^2 X, X) &= g_1(\phi X, \phi X) \\ &= \cos^2\theta(X) g_1(\varphi X, \varphi X) \\ &= \cos^2\theta(X) g_1(\varphi^2 X, X) \\ &= \cos^2\theta(X) g_1(X + \eta(X)\xi, X) \end{aligned} \tag{3.9}$$

for all $X \in \Gamma(ker\pi_*)$. Since g_1 is a Lorentzian metric, from (3.9) we have

$$\phi^2 X = \cos^2\theta(X)(X + \eta(X)\xi), \quad X \in \Gamma(ker\pi_*). \tag{3.10}$$

Let $\lambda = \cos^2\theta$. Then it is obvious that $\lambda \in [0, 1]$ and $\phi^2 = \lambda(I + \eta \otimes \xi)$.

Conversely, let us assume that there exist a constant $\lambda \in [0, 1]$ such that $\phi^2 = \lambda(I + \eta \otimes \xi)$ is satisfied. From (3.1) and (3.2) we get

$$\begin{aligned} \cos\theta(X) &= \frac{g_1(\varphi X, \phi X)}{\|\varphi X\| \|\phi X\|} \\ &= \frac{\lambda g_1(\varphi X, \varphi X)}{\|\varphi X\| \|\phi X\|}, \end{aligned}$$

for all $X \in \Gamma(ker\pi_*)$. Thus we have

$$cos\theta(X) = \frac{\lambda \|\varphi X\|}{\|\phi X\|}.$$

Since $cos\theta(X) = \frac{\|\phi X\|}{\|\varphi X\|}$, then by using the last equation we obtain $cos^2\theta(X) = \lambda$, which implies that $\theta(X)$ is a constant and π is a proper slant submersion. $\square$

From Theorem 3.8, (3.1) and (2.4) we have the following result.

Corollary 3.9. *Let π be a slant submersion from a Lorentzian almost paracontact manifold $(M_1, g_1, \varphi, \xi, \eta)$ onto a Riemannian manifold (M_2, g_2) with slant angle $\theta \in (0, 90)$. Then, for any $X, Y \in \Gamma(ker\pi_*)$, we have*

$$g_1(\phi X, \phi Y) = \cos^2\theta(g_1(X, Y) + \eta(X)\eta(Y)) \tag{3.11}$$

$$g_1(\omega X, \omega Y) = \sin^2\theta(g_1(X, Y) + \eta(X)\eta(Y)). \tag{3.12}$$

From (2.2), (3.1) and (3.3) we have the following result.

Corollary 3.10. *Let π be a slant submersion from a Lorentzian almost paracontact manifold $(M_1, g_1, \varphi, \xi, \eta)$ onto a Riemannian manifold (M_2, g_2) with slant angle $\theta \in (0, 90)$. Then, we have*

$$\phi^2 + B\omega = I + \eta \otimes \xi \tag{3.13}$$

$$\omega\phi + C\omega = 0. \tag{3.14}$$

Proposition 3.11. *Let π be a slant submersion from a Lorentzian almost paracontact manifold onto a Riemannian manifold with the slant angle $\theta \in (0, 90)$. If ω is parallel with respect to ∇ on $(ker\pi_*)$, then we have*

$$T_{\phi X}\phi X = \cos^2\theta(T_X X + \eta(X) T_X \xi) \tag{3.15}$$

for $X \in (ker\pi_*)$.

Proof. If ω is parallel, then from Lemma 3.7 we have $CT_XY = T_X\phi Y$ for $X, Y \in (ker\pi_*)$. Interchanging the role of X and Y, we get $CT_YX = T_Y\phi X$. Thus we have

$$CT_XY - CT_YX = T_X\phi Y - T_Y\phi X$$

Since T is symmetric, we derive $T_X\phi Y = T_Y\phi X$. Then substituting Y by ϕX we get $T_X\phi^2X = T_{\phi X}\phi X$. Finally using Theorem 3.8 we obtain (3.15).

We now investigate the geometry of the leaves of the distributions $(ker\pi_*)$ and $(ker\pi_*)^\perp$. ∎

Theorem 3.12. *Let π be a slant submersion from a Lorentzian almost paracontact manifold $(M_1, g_1, \varphi, \xi, \eta)$ onto a Riemannian manifold (M_2, g_2) with slant angle $\theta \in (0, 90)$. Then the distribution $(ker\pi_*)$ defines a totally geodesic foliation on M_1 if and only if*

$$-\eta(Y)g_1(T_X\xi, Z) = g_1(h\nabla_X\omega\phi Y, Z) + g_1(h\nabla_X\omega Y, CZ) + g_1(T_X\omega Y, BZ)$$

for $X, Y \in \Gamma(ker\pi_*)$ and $Z \in \Gamma((ker\pi_*)^\perp)$.

Proof. For $X, Y \in \Gamma(ker\pi_*)$ and $Z \in \Gamma((ker\pi_*)^\perp)$, from (2.4), (2.8) and (3.1) we have

$$g_1(\nabla_XY, Z) = g_1(\nabla_X\phi Y, \varphi Z) + g_1(\nabla_X\omega Y, \varphi Z).$$

Using (2.8), (3.1) and (3.3) we get

$$\begin{aligned} g_1(\nabla_XY, Z) &= g_1(\nabla_X\phi^2Y, Z) + g_1(\nabla_X\omega\phi Y, Z) \\ &+ g_1(\nabla_X\omega Y, BZ) + g_1(\nabla_X\omega Y, CZ). \end{aligned}$$

Then from (2.9) and Theorem 3.8 we obtain

$$\begin{aligned} g_1(\nabla_XY, Z) &= \cos^2\theta g_1(\nabla_XY, Z) + \eta(Y)g_1(T_X\xi, Z) + g_1(h\nabla_X\omega\phi Y, Z) \\ &+ g_1(T_X\omega Y, BZ) + g_1(h\nabla_X\omega Y, CZ). \end{aligned}$$

Hence we have

$$\begin{aligned} \sin^2\theta g_1(\nabla_XY, Z) &= \eta(Y)g_1(T_X\xi, Z) + g_1(h\nabla_X\omega\phi Y, Z) \\ &+ g_1(T_X\omega Y, BZ) + g_1(h\nabla_X\omega Y, CZ) \end{aligned}$$

which proves assertion. ∎

Theorem 3.13. *Let π be a slant submersion from a Lorentzian almost paracontact manifold $(M_1, g_1, \varphi, \xi, \eta)$ onto a Riemannian manifold (M_2, g_2) with slant angle $\theta \in (0, 90)$. Then the distribution $(ker\pi_*)^\perp$ defines a totally geodesic foliation on M_1 if and only if*

$$\phi(v\nabla_XBY + A_XCY) + B(A_XBY + h\nabla_XCY) = 0$$

for $X, Y \in \Gamma((ker\pi_*)^\perp)$.

Proof. For $X, Y \in \Gamma((ker\pi_*)^\perp)$, from (2.2), (2.8),(2.12),(2.13),(3.1) and (3.3) we get

$$\begin{aligned}\nabla_X Y &= \varphi\nabla_X\varphi Y = \varphi(\nabla_X BY + \nabla_X CY)\\ &= \varphi(A_X BY + v\nabla_X BY + A_X CY + h\nabla_X CY)\\ &= BA_X BY + CA_X BY + \phi v\nabla_X BY + \omega v\nabla_X BY + \phi A_X CY + \omega A_X CY\\ &+ Bh\nabla_X CY + Ch\nabla_X CY,\end{aligned}$$

which proves the assertion. □

Finally we give necessary and sufficient conditions for a slant submersion with slant angle $\theta \in (0, 90)$ to be totally geodesic. Recall that a differentiable map π between (semi-) Riemannian manifolds (M_1, g_1) and (M_2, g_2) is called a totally geodesic map if $(\nabla\pi_*)(X, Y) = 0$ for all $X, Y \in \Gamma(TM_1)$.

Theorem 3.14. *Let π be a slant submersion from a Lorentzian almost para-contact manifold $(M_1, g_1, \varphi, \xi, \eta)$ onto a Riemannian manifold (M_2, g_2) with slant angle $\theta \in (0, 90)$. Then π is totally geodesic if and only if*

$$-\cos^2\theta\eta(Y)g_1(T_X\xi, Z) = g_1(h\nabla_X\omega\phi Y, Z) + g_1(h\nabla_X\omega Y, CZ) + g_1(T_X\omega Y, CZ)$$

and

$$g_1(h\nabla_{Z_1}\omega\phi X, Z_2) = g_1(A_{Z_1}BZ_2 + h\nabla_{Z_1}CZ_2, \omega X) - \cos^2\theta\eta(X)g_1(A_{Z_1}\xi, Z_2)$$

for $Z, Z_1, Z_2 \in \Gamma((ker\pi_*)^\perp)$ and $X, Y \in \Gamma(ker\pi_*)$.

Proof. First of all, since π is a Lorentzian submersion we have

$$(\nabla\pi_*)(Z_1, Z_2) = 0$$

for $Z_1, Z_2 \in \Gamma((ker\pi_*)^\perp)$.
For $X, Y \in \Gamma(ker\pi_*)$ and $Z, Z_1, Z_2 \in \Gamma((ker\pi_*)^\perp)$, from (2.4),(2.8),(3.1) and (2.14) we have

$$g_2((\nabla\pi_*)(X, Y), \pi_*Z) = -g_1(\nabla_X\varphi\phi Y, Z) - g_1(\nabla_X\omega Y, \varphi Z).$$

Using (3.1) and (3.3) we get

$$\begin{aligned}g_2((\nabla\pi_*)(X, Y), \pi_*Z) &= -g_1(\nabla_X\phi^2 Y, Z) - g_1(\nabla_X\omega\phi Y, Z)\\ &- g_1(\nabla_X\omega Y, BZ) - g_1(\nabla_X\omega Y, CZ).\end{aligned}$$

Then Theorem 3.8 and (2.9) imply that

$$\begin{aligned}g_2((\nabla\pi_*)(X, Y), \pi_*Z) &= -\cos^2\theta g_1(\nabla_X Y, Z) - \cos^2\theta\eta(Y)g_1(T_X\xi, Z) - g_1(h\nabla_X\omega\phi Y, Z)\\ &- g_1(T_X\omega Y, BZ) - g_1(h\nabla_X\omega Y, CZ).\end{aligned}$$

Hence we obtain

$$\begin{aligned}\sin^2\theta g_2((\nabla\pi_*)(X, Y), \pi_*Z) &= -\cos^2\theta\eta(Y)g_1(T_X\xi, Z) - g_1(h\nabla_X\omega\phi Y, Z)\\ &- g_1(T_X\omega Y, BZ) - g_1(h\nabla_X\omega Y, CZ). \qquad (3.16)\end{aligned}$$

Similarly, we get

$$\begin{aligned}\sin^2\theta g_2((\nabla\pi_*)(X, Z_1), \pi_*Z_2) &= g_1(A_{Z_1}BZ_2 + h\nabla_{Z_1}CZ_2, \omega X) - \cos^2\theta\eta(X)g_1(A_{Z_1}\xi, Z_2)\\ &- g_1(h\nabla_{Z_1}\omega\phi X, Z_2). \qquad (3.17)\end{aligned}$$

Then the proof follows from (3.16) and (3.17). □

References

1. D. Allison, *Lorentzian Clairaut Submersions,* Geometriae Dedicata **63** (1996), 309-319.
2. B. Baird and J.C. Wood, Harmonic morphisms between Riemannian manifolds, Oxford science publications, 2003.
3. J.P. Bourguignon and H.B. Lawson, *A mathematician's visit to Kaluza- Klein theory,* Rend. Sem. Mat. Univ. Politec. Torino, Special Issue, (1989), 143-163.
4. J.P. Bourguignon and H.B. Lawson, *Stability and isolation phenomena for Yang-Mills fields,* Comm. Math. Phys. **79** (1981), 189-230.
5. A.V. Caldarella, *On paraquaternionic submersions between paraquaternionic Kähler manifolds,* Acta Applicandae Mathematicae **112** (2010),1-14.
6. D. Chinea, *Almost contact metric submersions,* Rend. Circ. Mat. Palermo, II Ser. **34** (1985), 89-104.
7. M. Falcitelli, S. Ianus and A.M. Pastore, Riemannian Submersions and Related Topics, World Scientific, 2004.
8. A. Gray, *Pseudo-Riemannian almost product manifolds and submersions,* J. Math. Mech. **16** (1967), 715-737.
9. Y. Gündüzalp, *Slant submersions from almost product Riemannian manifolds,* Turkish Journal of Mathematics **37** (2013), 863-873.
10. Y. Gündüzalp and B. Șahin, *Paracontact semi-Riemannian submersions,* Turkish Journal of Mathematics **37**(1)(2013), 114-128.
11. Y. Gündüzalp and B. Șahin, *Para-contact para-complex semi-Riemannian submersions,* Bull. Malays. Math. Sci. Soc. **37**(1) (2014), 139-152.
12. S. Ianus and M. Visinescu, *Kaluza-Klein theory with scalar fields and generalised Hopf manifolds,* Classical Quantum Gravity **4** (1987), 1317-1325.
13. S. Ianus and M. Visinescu, *Space-time compactification and Riemannian submersions,* The mathematical heritage of C.F. Gauss, World Sci. Publ., River Edge, NJ, (1991), 358-371.
14. M.T. Mustafa, *Applications of harmonic morphisms to gravity,* J. Math. phys. **41** (2000), 6918-6929.
15. S. Ianuș, R. Mazzocco and G.E. Vilcu, *Riemannian submersions from quaternionic manifolds,* Acta Appl. Math. **104** (2008), 83-89.
16. K. Matsumoto, *On Lorentzian Paracontact Manifolds,* Bull. Yamagata Univ. Nat. Sci. **12**(2) (1989), 151-156.
17. M.A. Khan, Khushwant Singh and V.A. Khan, *Slant submanifolds of LP-contact manifolds,* Differential Geometry-Dynamical Systems **12** (2010), 102-108.
18. B. O'Neill, *The fundamental equations of a submersion,* Michigan Math. J. **13** (1966), 459 469.
19. K.S. Park, *H-slant submersions,* Bull. Korean Math. Soc. **49** (2012), 329-338.
20. K.S. Park, *H-semi-invariant submersions,* Taiwanese Journal of Mathematics **16**(5) (2012), 1847-1863.
21. S. Prasad and R.H. Ojha, *Lorentzian paracontact submanifolds,* Publ. Math. Deprecen **44** (1994), 215-223.
22. B. Șahin, *Slant submersions from almost Hermitian manifolds,* Bull. Math. Soc.Sci. Math. Roumanie Tome **54**(102) (2011), 93-105.
23. B. Șahin, *Anti-invariant Riemannian submersions from almost Hermitian manifolds,* Cent. Eur. J. Math. **8**(3) (2010), 437-447.
24. B. Watson, *Almost Hermitian submersions,* J. Diff. Geom. **11** (1976), 147-165.
25. B. Watson, *G,G'-Riemannian submersions and nonlinear gauge field equations of general relativity,* Global analysis on manifolds Teubner-Texte Math. **57**,Teubner, Leipzig, (1983), 324-249.

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