Gulf Journal of Mathematics
Vol 3, Issue 2 (2015) 61-65

THE STRONGLY HOPFIAN ABELIAN GROUPS

S. ABDELALIM[2], A. CHILLALI[1*] AND H. ESSANOUNI[3]

ABSTRACT. Let A be an abelian group, A is said strongly hopfian if for all $f \in End(A)$ we have the chain $Ker(f) \subseteq Ker(f^2) \subseteq Ker(f^3) \subseteq Ker(f^4) \subseteq \cdots$ is stationary. In this paper, we characterize strongly hopfian abelian group in the class of torsion abelian group. After we show that the p-component of strongly hopfian abelian group is strongly hopfian. But for the torsion part, we construct strongly hopfian abelian group whose the torsion part is not strongly hopfian.

1. Introduction

Let A be an abelian group. A is said strongly hopfian, if A satisfies the following property:

(i) $\forall f \in End(A)$, $\exists n \in \mathbb{N}^*$ such that $Ker(f^n) \cap f(A) = \{0\}$.
(ii) For all $f \in End(A)$, the chain $Ker(f) \subseteq Ker(f^2) \subseteq Ker(f^3) \subseteq \cdots$ is stationary.
(iii) $\forall f \in End(A)$ $\exists n \in \mathbb{N}^*$ such that $Ker(f^n) = Ker(f^{n+1})$.

In [3] A. Hmaimou, A. Kaidi, E. Sanchez study the strongly Hopfian (respectively strongly co-hopfian) modules in Certain Class of modules, and the authors showed the equivalence between strongly hopfian and hopfian for a module in a semi-simple ring. So the question is what are the properties of strongly hopfian abelian groups ?
In this paper we will Characterize the strongly hopfian abelian groups in class of torsion abelian groups and we will show that the p-component of strongly hopfian abelian groups is also strongly hopfian but for the torsion portion we construct strongly hopfian abelian group whose the torsion portion is not strongly hopfian.
For any abelian group we have:

Noetherian $\Longrightarrow$ strongly hopfian [3] $\Longrightarrow$ hopfian [4] $\Longrightarrow$ Generalize hopfian [2],

but the reverse implications are not always correct.

2. Strongly Hopfian Abelian Groups in the Category of Torsion Abelian Groups

In this section, we will determine the structure of strongly hopfian torsion abelian groups.

Date: Received: Feb 28, 2015.
* Corresponding author.
2010 *Mathematics Subject Classification.* 20K30, 20K40, 20K27.
Key words and phrases. Abelian Group, Strongly Hopfian, Chain Groups, Torsion.

Theorem 2.1. *Let A a torsion group then the following properties are equivalents:*

1- *A is strongly hopfian.*

2- *There exists $n_0 \in \mathbb{N}^*$ such that for all prime number p, we have $p^n A_p = 0$ and $\mid A_p \mid \leq p^n n$ with A_p is the p-component of A.*

Proof. **(i)** $\Longrightarrow$ **(ii)** Let $\alpha \in End(A)$ defined by :

$$\begin{array}{rccc} \alpha : & A = \oplus_p A_p & \longrightarrow & A \\ & \sum\limits_p x_p & \longmapsto & \sum\limits_p p x_p \end{array}$$

According on the assumption, $\exists n_1 \in \mathbb{N}^*$ such that $Ker(\alpha^{n_1}) = Ker(\alpha^n)$ for all $n > n_1$. Show that $p^{n_1} A_p = 0$. **(a)** Let $x \in A_p$, we pose $o(x) = p^{n_0}$.Suppose that $n_0 > n_1$. It is clearly that $Ker(\alpha^{n_1}) = Ker(\alpha^{n_0})$. We can write $\alpha^{n_0}(x) = p^{n_0} x = 0$, then $x \in Ker(\alpha^{n_1}) = Ker(\alpha^{n_0})$, ie, $p^{n_1} x = \alpha^{n_1}(x) = 0$. So, $p^{n_1} A_p = 0$. By (17.2 page 88 [1]) A_p is direct sums of cyclic groups. Suppose that $A_p = \oplus_{i \in I_p} < x_{p,i} >$ and show that $\mid I_p \mid < \infty$, suppose the contrary, then $A_p = \oplus_{i \in \mathbb{N}^*} < x_{p,k} > \oplus_{i \in I'_p} < x_{p,i} >$ who $o(x_{p,k}) \leq o(x_{p,k+1})$. By (27.5 page 118 [1]), we have $A = A_p \oplus A'$. Let $f \in End(A)$ defined by:

$$\begin{array}{rccc} f : & A & \longrightarrow & A \\ & \sum\limits_{k=1}^{n_0} m_k x_{p,k} + \sum\limits_{k=1}^{n'} d_k x_{p,i_k} + a' & \longmapsto & \sum\limits_{k=1}^{n_0-1} m_{k+1} x_{p,k} + \sum\limits_{k=1}^{n'} d_k x_{p,i_k} + a' \end{array}$$

It's clearly that $Ker(f) = < x_{p,1} >$, $Ker(f^2) = < x_{p,1} > \oplus < x_{p,2} >$,...,
$Ker(f^n) = < x_{p,1} > \oplus < x_{p,2} > \oplus ... \oplus < x_{p,n} >$, therefore the sequence $Ker(f^n)$ isn't stationary which is absurd, then I_p is finite. We pose $A_p = \oplus_{k=1}^{n_p} < x_{p,k} >$.
We will show that the set $\{n_p /$ `p is prime number` $\}$ is bounded. We pose α_p the endomorphism of A_p defined by:

$$\begin{array}{rccc} \alpha_p : & A_p & \longrightarrow & A_p \\ & \sum\limits_{k=1}^{n_p} m_k x_{p,k} & \longmapsto & \sum\limits_{k=1}^{n_p-1} m_{k+1} x_{p,k} \end{array}$$

Since, $A = \oplus_p A_p$. We pose $\alpha = \oplus_p \alpha_p$, we can write: $Ker(\alpha^n) = \oplus p Ker(\alpha_p^n)$ **(1)**.

According on the assumption, $\exists n' \in \mathbb{N}^*$ such that $Ker(\alpha^{n'}) = Ker(\alpha^{n'+1})$ **(2)**. Suppose there exists a prime number p such $n_p > n'$, by **(1)** and **(2)** we have $Ker(\alpha_p^{n'}) = Ker(\alpha_p^{n'+1})$. It clearly that $Ker(\alpha_p^{n'}) = \oplus_{k=1}^{n'} < x_{p,k} >$ and $Ker(\alpha_p^{n'+1}) = \oplus_{k=1}^{n'+1} < x_{p,k} >$ then $Ker(\alpha_p^{n'}) \neq Ker(\alpha_p^{n'+1})$. We pose $m_0 = \sup$ $\{n_p /$ p prime number $\}$ and $n^* = n_1 + m_0$. By **(a)** we have $p^{n^*} A_p = 0$ for all p prime number. Then $A_p = \oplus_{k=1}^{n_p} < xp, k >$. Since $O(x_{p,k}) < p^{n^*}$ and $n_p \leq n^*$ then $\mid A_p \mid \leq n^* p^{n^*}$.

(ii) $\Longrightarrow$ **(i)** By **(ii)**, there exists $n_0 \in \mathbb{N}^*$ such that $\mid A_p \mid < p^{n_0}$ for all p prime number. We consider the restriction α_p of α on A_p. It clearly that $Ker(\alpha^n) = \oplus_p Ker(\alpha_p^n)$. Now to show that: $Ker(\alpha^{2n_0}) = Ker(\alpha^{2n_0+1})$.
Assume the contrary, then $Ker(\alpha_p) \subset Ker(\alpha_p^2) \subset Ker(\alpha_p^3) ... \subset Ker(\alpha_p^{2n_0}) \subset Ker(\alpha_p^{2n_0+1})$ we have: $Ker(\alpha_p^k) \subset Ker(\alpha_p^{k+1})$ then the quotient group $Ker(\alpha_p^{k+1}) / Ker(\alpha_p^k)$ is not reduced to zero. Then $\mid Ker(\alpha_p^{k+1}) \mid \geq p \mid Ker(\alpha_p) \mid$. So, $p^{2n_0+1} \leq \mid Ker(\alpha_p^{2n_0+1}) \mid \leq \mid A_p \mid \leq p^{n_0} n_0 \leq p^{2n_0}$ which is absurd. Finally, $Ker(\alpha_p^{2n_0}) = Ker(\alpha_p^{2n_0+1})$. $\square$

Corollary 2.2. *Any subgroup of strongly hopfian torsion abelian group is strongly hopfian.*

Proposition 2.3. *A direct factor of strongly hopfian abelian group is strongly hopfian.*

Proof. Let B direct factor of strongly hopfian abelian group of A. We pose $A = B \oplus C$ and $\alpha_1 = \alpha + id_C$ where $\alpha \in End(B)$. It is clearly to see that $\alpha_1 \in End(A)$ and $Ker(\alpha_1^n) = Ker(\alpha^n)$. According on the assumption $\exists n \in \mathbb{N}$ such that $Ker(\alpha_1^n) = Ker(\alpha_1^{n+1})$ then $Ker(\alpha^n) = Ker(\alpha^{n+1})$. □

Proposition 2.4. *The direct sum of two strongly hopfian torsion abelian groups is strongly hopfian.*

Proof. Let A and B two strongly hopfian torsion abelian groups then $\exists n_0, n_1 \in \mathbb{N}^*$ such that $\mid A_p \mid < n_0 p^{n_0}$, $\mid B_p \mid < n_1 p^{n_1}$ **(1)** and $T = A \oplus B$ then $T_p = A_p \oplus B_p$.We pose $n_2 = n_0 + n_1$, by **(1)** $\mid T_p \mid = \mid A_p \mid \times \mid B_p \mid < n_0 \times n_1 p^{n_0+n_1} < n_2 p^{n_2}$ and $p^{n_2} T_p = 0$ hence T is strongly hopfian. □

Proposition 2.5. *Let A and B two strongly hopfian abelian groups with $Hom(A, B) = 0$ then $A \oplus B$ is strongly hopfian.*

Proof. Let $f \in End(A \oplus B)$, p_1 the projection from $A \oplus B$ to A and p_2 the projection from $A \oplus B$ to B, we consider f_1 the restriction of f on A and f_2 the restriction of $p_2 f$ on B then $\exists n_0 \in \mathbb{N}^*$ such that $Ker(f_1^{n_0}) = Ker(f_1^{n_0+1})$ and $Ker(f_2^{n_0}) = Ker(f_2^{n_0+1})$ **(a)**and since $Hom(A, B) = 0$ then $f_2^n = p_2 f^n / B$ **(1)**. For to finish the proof, we show that $Ker(f^{2n_0}) = Ker(f^{4n_0})$. Let $x \in Ker(f^{4n_0})$ then $x = x_1 + x_2$ with $x_1 \in A$ and $x_2 \in B$, then $p_2 f^{4n_0}(x_2) = (p_2 f)^{4n_0}(x_2) = (p_2 f)^{n_0}(x_2) = p_2 f^{n_0}(x_2)$ which is implies $f^{n_0}(x_2) \in A$. And since $f^{n_0}(x_1) \in A$ because we have $Hom(A, B) = 0$ then $f^{n_0}(x_1 + x_2) \in A$. We note that $X = f^{n_0}(x_1 + x_2)$**(2)**, it is clearly to see that $f_1^{3n_0}(X) = f^{4n_0}(x_1 + x_2) = f^{4n_0}(x) = 0$ then by **(a)** we have $X \in Ker(f_1^{3n_0}) = Ker(f_1^{3n_0})$ and since $X = f^{n_0}(x)$ then $f^{2n_0}(x) = f_1^{n_0}(X)$ therefore $x \in Ker(f^{2n_0})$. □

Corollary 2.6. *If A and B are two strongly hopfian abelian groups such that $Hom(A, B) = Hom(B, A) = 0$ then $A \oplus B$ is strongly hopfian.*

Proposition 2.7. *Let A a strongly hopfian abelian group and p is a prime number then the p-component of A is strongly hopfian.*

Proof. We pose $f = pid$ then $f^n = p^n id$. It is clearly to see that: $Ker(f^n) = A[p^n]$. According on the assumption $\exists n_0 \in \mathbb{N}^*$ such that $Ker(f^{n_0}) = Ker(f^n)$ for all $n > n_0$. Then $A[p^n] = A[p^{n_0}]\ \forall n > n_0$. We can deduce that $A[p^{n_0}]$ where A_p is the p-component of A which is bounded. By (27.4 page 118 [1]), A_p is a direct factor of A then A_p is strongly hopfian. It follows from theorem 2.1 that p-component of A is finite. □

Remark 2.8. The p-component of a strongly hopfian abelian group is strongly hopfian.

3. Construction of a strongly hopfian abelian group whose the torsion part is not strongly hopfian

In the previous section we saw that the p-component of a strongly hopfian abelian group is strongly hopfian, so it is natural to ask the same question for the torsion part. In this sense we will build a strongly abelian group whose the torsion portion is not strongly hopfian.

Let G a group of products cyclic groups: $G = \prod_{k \in \mathbb{N}} < x_k >$ where $o(x_k) = p_k^k$, $o(x_0) = p_0$ and $P = \{p_k / k \in \mathbb{N}\}$ the set of all distinct primes. Considering the projection φ_k of $\prod_{n \geq 0} < x_n >$ on $< x_k >$. For $m \geq 1$, we define the element $b_m \in \prod_{n \geq 0} < x_n >$ by:

$$\varphi_n(b_m) = \begin{cases} 0 & \texttt{if} \ m > n \\ y_{m,n} & \texttt{if} \ m \leq n \end{cases}$$

and that the sequence $(y_{m,n})_{m\leq n}$ satisfies the following condition: $y_{0,n} = x_n \ \forall n \in \mathbb{N}$, $p_0 y_{1,n} = y_{0,n} \ \forall n \in \mathbb{N}^*$ and $(p_0...p_{m-1})^m y_{m,n} = y_{m-1,n}$ for all $n \geq m$ **(1)** we have: $(p_0...p_k)^{k+1} b_{k+1} = b_k - y_{k,k}$ **(2)**
then $(p_0...p_k)^{k+1}(p_0...p_{k-1})^k b_{k+1} = b_{k-1} - y_{k-1,k-1} - y_{k-1,k}$
and then $(p_0...p_k)^{k+1}(p_0...p_{k-1})^k...(p_1p_2)^2 p_0 b_{k+1} = b_0 - x_0 - x_1 - ... - x_k$
which is implies that: $p_0^{\frac{(k+2)(k+1)}{2}} p_1^{\frac{(k+2)(k+1)}{2}-1} ... p_i^{\frac{(k+2)(k+1)}{2}-\frac{(i+1)(i)}{2}} ... p_k^{\frac{(k+2)(k+1)}{2}-\frac{(k+1)(k)}{2}} b_{k+1} = b_0 - x_0 - x_1 - ... - x_k$ **(3)**. Let A the subgroup of $\prod_{k\in\mathbb{N}} < x_k >$ generated by $\{x_k/k \in \mathbb{N}\} \cup \{b_n/n \in \mathbb{N}\}$ **(D_1)**. Show that A is Strongly hopfian. Let $f \in End(A)$, we can write $f(b_0) = \sum_{k=0}^{n_0} m_k x_k + n_k b_k$ and we pose that:

$$M_{k+1} = p_0^{\frac{(k+2)(k+1)}{2}} p_1^{\frac{(k+2)(k+1)}{2}-1} ... p_i^{\frac{(k+2)(k+1)}{2}-\frac{(i+1)(i)}{2}} ... p_k^{\frac{(k+2)(k+1)}{2}-\frac{(k+1)(k)}{2}}, \forall k \in \mathbb{N}^* \textbf{(4)}.$$

By **(3)** we have $f(Mb_0) = (\sum_{k=0}^{n_0} m_k^{'} x_k) + Nb_0$ then $f(p_0p_1p_2^2...p_{n_0}^{n_0} Mb_0) = p_0p_1p_2^2...p_{n_0}^{n_0} Nb_0$. If we pose $M^{'} = p_0p_1p_2^2...p_{n_0}^{n_0} M$ and $N^{'} = p_0p_1p_2^2...p_{n_0}^{n_0} N$, it's clear that $f(M^{'}b_0) = N^{'}b_0$ **(5)** and by **(3)** $x_0 + x_1 + ... + x_k + M_{k+1}b_{k+1} = b_0$. And thanks to **(4)** we deduct p_k^k divides M_{k+1}, so we can write $M_{k+1} = p_k^k M_{k+1}^{'}$ and by **(3), (4), (5) and (6)** we have: $f(M^{'}b_0) = f(M^{'}(x_0 + x_1 + ... + x_k + M_{k+1}b_{k+1})) = f(M^{'}(x_0 + x_1 + ... + x_k + p_k^k M_{k+1}^{'} b_{k+1})) = f(M^{'}x_0 + M^{'}x_1 + ... + M^{'}x_k + p_k^k M^{'} M_{k+1}^{'} b_{k+1})$ Then $\varphi_k f(M^{'}b_0) = \varphi_k(f(M^{'}x_0 + M^{'}x_1 + ... + M^{'}x_k)) + p_k^k \varphi_k f(M^{'}M_{k+1}^{'}b_0) = f(M^{'}x_k) = N^{'}x_k \ \forall k \in \mathbb{N}^*$ **(D_2)**. We denote T the part of the torsion of G, show that $T = \oplus_{n\geq 0} < x_n >$. Let $X \in T$ then $\exists M \in \mathbb{N}^*$ such that $MX = 0$, then there exists $k^{''} \in \mathbb{N}^*$ such that p_k does not divide M for all $k > k^{''}$ **(7)**. We Pose $\varphi(X) = X_k$ then $MX_k = \varphi(MX) = 0$ and by **(7)** $X_k = 0$, therefore $X \in \oplus_{n\geq 0} < x_n >$. We denote T_A the part of the torsion of A, clearly $\oplus_{n\geq 0} < x_n > \subset T_A \subset T \subset \oplus_{n\geq 0} < x_n >$ thus $T_A = \oplus_{k\geq 0} < x_k >$ (F_0).
Show that $Ker(f) \subset \oplus_{k\geq 0} < x_k >$. Let $X \in Ker(f)$ then $\exists M, N^{''} \in \mathbb{N}^*$ such that $MX = N^{''}b_0$. Suppose that $N^{''}b_0 \neq 0$ then $\exists k^* \in \mathbb{N}$ such that p_k is prime with (M, $N^{''}$, $M^{'}$ and $N^{'}$) $\forall k > k^*$ **(8)** and thanks to bf (3) and (4), we have:

$N^{''}b_0 = N^{''}x_0 + N^{''}x_1 + ... + N^{''}x_k + N^{''}M_{k+1}b_{k+1} = M^{'}N^{''}x_0 + M^{'}N^{''}x_1 + ... + M^{'}N^{''}x_k + M^{'}N^{''}M_{k+1}b_{k+1} = M^{'}N^{''}x_0 + M^{'}N^{''}x_1 + ... + M^{'}N^{''}x_k + M^{'}N^{''}p_k^k M_{k+1}^{'} b_{k+1}$
So, $0 = \varphi_k f(MX) = \varphi_k f(M^{'}N^{''}b_0) = f(M^{'}N^{''}x_k) = N^{''}N^{'}x_k$ then p_k divided $N^{''}N^{'}$ this is contradictory with **(8)** then $N^{''}b_0 = 0$. So, $Ker(f) \subset \oplus_{k\geq 0} < x_k >$. By **(D_2)** $f(M^{'}x_k) = N^{'}x_k$, $\forall k \in \mathbb{N}^*$, then $\exists k_0 \in \mathbb{N}^*$ such that p_k is primer with ($M^{'}$ and $N^{'}$) for all $k > k_0$ **(F_1)**.
Show that $Ker(f) \subset \oplus_{k\geq 0}^{k_0} < x_k >$. Let $X \in Ker(f)$ then $X \in \oplus_{k\geq 0} < x_k >$ i.e, $0 = f(M^{'}X) = \sum_{k=1}^{n_0} f(M^{'}m_k x_k) = \sum_{k=1}^{n_0} N^{'}m_k x_k$. And since $N^{'}$ is primer with $p_k, \forall k > k_0$ then $Ker(f) \subset \oplus_{k\geq 0}^{k_0} < x_k >$. By recurrence we will show that $Ker(f^n) \subset \oplus_{k\geq 0}^{k_0} < x_k >$, $\forall n > 0$. Suppose that $Ker(f^n) \subset \oplus_{k\geq 0}^{k_0} < x_k >$ **(9)**, let $X \in Ker(f^{n+1})$ then $f(X) \in Ker(f^n)$ **(10)**. By **(9)** $f(X) \in \oplus_{k\geq 0}^{k_0} < x_k >$ then $\exists M_0 \in \mathbb{N}^*$ such that $M_0 f(X) = 0$ then $f(M_0X) = 0$ and $X \in T_A$. Finally we pose $X = \sum_{k=0}^{k_0} m_k x_k + \sum_{k=k_0+1}^{n_0} m_k x_k$. By

(D_2) we have:

$$f(M^{'}X)=\sum_{k=0}^{k_0} m_k f(M^{'}x_k)+\sum_{k=k_0+1}^{n_0} m_k f(M^{'}x_k)=\sum_{k=0}^{k_0} m_k N^{'}x_k+\sum_{k=k_0+1}^{n_0} m_k N^{'}x_k$$

And by **(9)** and **(10)**, we have: $\sum\limits_{k=k_0+1}^{n_0} m_k N^{'}x_k = 0$. Such as $N^{'}$ is primer with p_k, for all $k > k_0$, (voir (F_1)), from where $X = \sum\limits_{k=0}^{k_0} m_k x_k$ and therefore $Ker(f^n) \subset \oplus_{k\geq 0}^{k_0} < x_k >$ and as $\mid \oplus_{k\geq 0}^{k_0} < x_k > \mid < \infty$ then the next sequence $Ker(f) \subset Ker(f^2) \subset Ker(f^2) \subset ...Ker(f^n) \subset Ker(f^{n+1}) \subset ...$ is stationary . This shows that A is strongly hopfian. The tortion part T_A of A is $\oplus_{k\geq 0} < x_k >$ with $o(x_k) = p^k$ see(F_0). By (theorem2.1)T_A is not strongly hopfian.

4. Conclusion

In this work, we have extended the results of strongly hopfian abelian group, we characterize strongly hopfian abelian group in the class of torsion abelian group. We construct strongly hopfian abelian group whose the torsion part is not strongly hopfian.

References

1. L. Fuchs; *Infinite Abelian Groups*, Academic press New York,vol. 1 (1970).
2. A. Haghany and M.R. Vedadi; *Generalized Hopfian Modules*, Journal of Algebra (2002), 324-341.
3. A. Hmaimou, A. Kaidi, E. Sànchez Campos; *Generalized Fitting modules and rings*, Journal of Algebra 308, (2007) 199-244.
4. K. Varadarjran; *Some Recent results on Hopficity on co-Hopficity and related properties, in International Symposium on ring theory*, Ed. by F. Berkenmeier, Jae Keol Park and Young Soo Park, Birkhauser (2000), 371-392.

[1]Department of Mathematics, Physics and Computing, LSI, FPT, Taza, Box 1223, University S.M. Ben Abdellah Fez, Morocco.
E-mail address: abdelhakim.chillali@usmba.ac.ma

[2]Department of Mathematical and Computer Sciences, Labo TAGESUP, Faculty of Sciences Ain Choc, University of Hassan II Casablanca, Morocco.
E-mail address: seddikabd@hotmail.com

[3]Department of Mathematical and Computer Sciences, Faculty of Sciences,University of Mohamed V Agdal, BP.1014, Rabat, Morocco.