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ON GORENSTEIN GLOBAL DIMENSION OF TENSOR PRODUCT OF ALGEBRAS OVER A FIELD

NAJIB MAHDOU[1*] AND MOHAMMED TAMEKKANTE[2]

Abstract. In this paper, we study the Gorenstein global dimension of the tensor product of associative algebras. In particular, the Gorenstein global dimension of the enveloping algebras A^e of the algebras A.

1. Introduction

Throughout this note all rings and algebras considered are commutative with identity elements, and all ring homomorphisms are unital. In addition, k stands for a field.

Let R be a ring, and let M be an R-module. As usual we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$ and $\mathrm{fd}_R(M)$ to denote, respectively, the classical projective dimension, injective dimension and flat dimension of M. By $\mathrm{gldim}(R)$ and $\mathrm{wdim}(R)$ we denote, respectively, the classical global dimension and weak dimension of R.

For a Noetherian ring R, Auslander and Bridger [1] introduced the G-dimension, $\mathrm{Gdim}_R(M)$, for every finitely generated R-module M. They showed that there is an inequality $\mathrm{Gdim}_R(M) \leq \mathrm{pd}_R(M)$ for all finite R-modules M, and equality holds if $\mathrm{pd}_R(M)$ is finite.

Several decades later, Enochs and Jenda [10] defined the notion of Gorenstein projective dimension (G-projective dimension for short), as an extension of G-dimension to modules that are not necessarily finitely generated, and the Gorenstein injective dimension (G-injective dimension for short) as a dual notion of Gorenstein projective dimension. Then, to complete the analogy with the classical homological dimension, Enochs, Jenda and Torrecillas [11] introduced the Gorenstein flat dimension. Some references are [2, 3, 7, 8, 9, 10, 11, 15].

Recently in [3], the authors study of global Gorenstein dimensions of rings, which are called Gorenstein global projective (resp. injective, weak) dimension of R, denoted by $\mathrm{GPD}\,(R)$ (resp. $GID(R)$, $G.wdim(R)$), and, respectively, defined as follows:

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* Corresponding author.

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$$\begin{array}{llll} 1) & \mathrm{GPD}\,(R) & = & sup\{\mathrm{Gpd}_R(M) \mid M\ R-module\} \\ 2) & \mathrm{GID}\,(R) & = & sup\{\mathrm{Gid}_R(M) \mid M\ R-module\} \\ 3) & \mathrm{Gwdim}\,(R) & = & sup\{\mathrm{Gfd}_R(M) \mid M\ R-module\} \end{array}$$

They showed that $\mathrm{Gwdim}\,(R) \leq \mathrm{GID}\,(R) = \mathrm{GPD}\,(R)$ ([3, Theorems 1.1 and Corollary 1.2(1)]). So, according to the terminology of the classical theory of homological dimensions of rings, the common value of $\mathrm{GPD}\,(R)$ and $\mathrm{GID}\,(R)$ is called Gorenstein global dimension of R, and denoted by $\mathrm{Ggldim}\,(R)$.
They also proved that the Gorenstein global (resp. weak) dimension is a refinement of the classical global (resp. weak) dimension of rings. That is : $\mathrm{Ggldim}\,(R) \leq \mathrm{gldim}(R)$ (resp. $\mathrm{Gwdim}\,(R) \leq \mathrm{wdim}(R)$) with equality if $\mathrm{gldim}(R)$ (resp. $\mathrm{wdim}(R)$) is finite ([3, Corollary 1.2(2 and 3)]).

There are several attempts to investigate which properties are conserved under tensor product operations. For example, it is well-known that the tensor product $R \otimes_k S$ of regular rings is not regular in general, even if we assume R and S are k–algebra, see [18, Remark 1.7]. In [16], Watanabe, Ishikawa, Tachibana, and Otsuka, showed that under a suitable condition tensor products of regular rings are complete intersections. In [13] it is proved that the tensor product $R \otimes_k S$ of Cohen–Macaulay rings are again Cohen–Macaulay if we assume R is flat A–module and S is a finitely generated k–module, and in [16], it is shown that the same is true for Gorenstein rings. Recently, in [5], Bouchiba and Kabbaj showed that if R and S are k–algebras such that $R \otimes_k S$ is Noetherian, then $R \otimes_k S$ is a Cohen–Macaulay ring if and only if R and S are Cohen–Macaulay rings. In [18, Theorem 1.6] the authors showed that if R and S be non–zero k–algebras such that $R \otimes_k S$ is Noetherian, then $R \otimes_k S$ is locally complete intersection (resp. Gorenstein, Cohen–Macaulay) if and only if R and S are locally complete intersection (resp. Gorenstein, Cohen–Macaulay).

Since regular (resp. Gorenstein) rings are characterized by finiteness of global (Gorenstein global) dimension, so it is natural to see whether the finiteness of these dimensions conserved under tensor product operations. In this way, Auslander studied global dimension of the tensor product of two algebras. In this paper we study Gorenstein global dimension of two algebras over a field. We show that for two k-algebras R and S if they have finite Gorenstein global dimensions, then $\mathrm{Ggldim}\,(R \otimes_k S) \geq \mathrm{Ggldim}\,(R) + \mathrm{Ggldim}\,(S)$. This shows that $R \otimes_k S$ is $n+m$-Gorenstein if and only if R is n-Gorenstein and S is m-Gorenstein.

Let S^* be the opposite algebra of S. Then the k-algebra $S \otimes_k S^*$ is called the envelope algebra of S and it is denoted by S^e. In the last part of this paper we will show that for two finitely generated k-algebras R and S, $\mathrm{Gpd}_{(R\otimes S)^e}(R\otimes S) \geq \mathrm{Gpd}_{R^e}(R) + \mathrm{Gpd}_{S^e}(S)$ provided $\mathrm{Gpd}_{R^e}(R)$ and $\mathrm{Gpd}_{S^e}(S)$ are finite.

2. Main results

Let k be a commutative ring (with a unit element denoted by 1). Given two k-algebras S and T, the tensor product $S \otimes_k T$ is a k-module, and the multiplication

$$(s_1 \otimes t_1)(s_2 \otimes t_2) := (s_1 s_2 \otimes t_1 t_2)$$

converts $S \otimes_k T$ into a k-algebra. As long as the ring k is fixed we shall frequently write $S \otimes T$ for $S \otimes_k T$.

Lemma 2.1. *Let R be a ring with finite Gorenstein global dimension, then, for a positive integer n, the following are equivalent:*

1. *$Ggldim(R) \leq n$;*
2. *$Gpd_R(M) \leq n$ for all finitely generated module M;*
3. *$\mathrm{pd}(E) \leq n$ for every injective module E.*

Proof. The implication $(1 \Rightarrow 2)$ is obvious. Conversely if we suppose that $\mathrm{Gpd}_R(M) \leq n$ for all finitely generated module M. Immediately for any ideal I of R we have $\mathrm{Gpd}_R(R/I) \leq n$. Thus, from [15, Theorem 2.20], for any projective module P we have $Ext^i(R/I, P) = 0$ for all $i > n$. Hence, by [17, Lemma 9.11], $\mathrm{id}(P) \leq n$. Then, for any module M and any projective module P, we have $\mathrm{Ext}^i(M, P) = 0$ for all $i > n$. Therefore, by [15, Theorem 2.20], $\mathrm{Ggldim}(R) = sup\{\mathrm{Gpd}(M) | M \ an\ R-module\} \leq n$.
Now we show the equivalence $(1 \Leftrightarrow 3)$. Note that $\mathrm{Ggldim}(R) = sup\{\mathrm{Gid}(M) | M\ an\ R-module\}$ (by [3, Theorem 1.1]). Thus, using [15, Theorem 2.22], $\mathrm{Ggldim}(R) \leq n \Leftrightarrow \mathrm{Ext}^i(E, M) = 0$ for each $i > n$ and for any injective module E and for any R-module M. Thus, $\mathrm{Ggldim}(R) \leq n \Leftrightarrow \mathrm{pd}(E) \leq n$ for any injective module E, as desired. □

Recall that a ring R is said to be n-Gorenstein (for some $n \geq 0$), if R is Noetherian and $\mathrm{id}_R(R) \leq n$. Also R is said to be Gorenstein if it is n-Gorenstein for some positive integer n ([14]). We have:

Proposition 2.2. *If R is Noetherian, then for a positive integer n, the following are equivalent:*

1. *R is n-Gorenstein;*
2. *$Ggldim(R) \leq n$.*

Proof. The proof is deduced from [12] and Lemma 2.1. □

Proposition 2.3. *Let S and T be two algebras over a semisimple ring k. If A is a projective S-module and B is a Gorenstein projective T-module, then $A \otimes_k B$ is a Gorenstein projective $S \otimes T$-module.*

Proof. Consider a complete projective resolution of T-modules:

$$\mathbf{P} = \cdots \to P_{-1} \overset{d_{-1}}{\to} P_0 \overset{d_0}{\to} P_1 \overset{d_1}{\to} P_2 \overset{d_2}{\to} \cdots .$$

such that $B = Im(d_0)$ (see [15, Definition 2.1]). Note that all morphisms d_i are also a k-morphisms. Indeed, for any $\lambda \in k$ and $x \in P_i$, we have

$$d_i(\lambda.x) = d_i(\lambda.1x) = (\lambda.1)d_i(x) = \lambda.d_i(x)$$

The k-module A is projective since k is semisimple. Now applying $A \otimes_k -$ to $\mathbf{P}$ we obtain the exact sequence of k-modules:

$$A \otimes_k \mathbf{P} = \ldots \to A \otimes_k P_{-1} \to A \otimes_k P_0 \to A \otimes_k P_1 \to A \otimes_k P_2 \to \ldots$$

Moreover, for each $i \in \mathbb{Z}$,

$$1_A \otimes d_i((t \otimes s)(a \otimes p)) = 1_A \otimes d_i(ta \otimes sp) = ta \otimes d_i(sp) = ta \otimes sd_i(p) = (t \otimes s)(1_A \otimes d_i(a \otimes p))$$

Thus the above exact sequence is also an $S \otimes T$ sequence. By [6, IX, Corollary 2.5], $S \otimes T$-modules $A \otimes_k P_i$ is projective for each i. Moreover, from [6, IX, Corollary 2.2], for any projective $S \otimes T$-module Q we have,

$$\mathrm{Hom}_{S \otimes T}(A \otimes_k \mathbf{P}, Q) \cong \mathrm{Hom}_S(A, \mathrm{Hom}_T(\mathbf{P}, Q))$$

But Q is also a projective T-module by [6, IX, Corollary 2.4]. Hence, $\mathrm{Hom}_T(\mathbf{P}, Q)$ is exact. So, $\mathrm{Hom}_{S \otimes T}(A \otimes_k \mathbf{P}, Q)$ is exact. Consequently, $A \otimes_k B = Im(1_A \otimes d_0)$ is a Gorenstein projective $S \otimes T$-module. □

Let S and T be two k-algebras. It is proved in [16, Theorem 2] that the tensor product $S \otimes T$ of Gorenstein rings is again Gorenstein if we assume S is flat k-module and T is a finitely generated k-module. For other conditions to obtain this conclusion we can see also [16, Part II]. Now we compare the Gorenstein global dimension of Noetherian k-Algebras (where k is a field) S and T with this of there tensor product in the general case.

Theorem 2.4. *Let S and T be two Noetherian k-algebras over a field k with finite Gorenstein global dimensions. Then, $Ggldim\,(S \otimes T) \geq Ggldim\,(S) + Ggldim\,(T)$.*

Lemma 2.5. *Let S and T be two Noetherian algebras over a field k. If A (resp. B) is a finitely generated S-module (resp. T-module) with finite Gorenstein projective dimension, then $Gpd_{S \otimes T}(A \otimes_k B) \geq Gpd_S(A) + Gpd_T(B)$.*

Proof. Set $\mathrm{Gpd}_S(A) = n$ and $\mathrm{Gpd}_T(B) = m$. From [15, Theorem 2.20], there is a projective S-module P (resp. T-module Q) such that $\mathrm{Ext}^n_S(A, P) \neq 0$ and $\mathrm{Ext}^m_T(B, Q) \neq 0$. Then, using [6, XI, Theorem 3.1] we conclude that $\mathrm{Ext}^{n+m}_{S \otimes T}(A \otimes_k B, P \otimes_k Q) \neq 0$. On the other hand, $P \otimes_k Q$ is a projective $S \otimes T$-module (by [6, IX, Corollary 2.5]). Then, again by [15, Theorem 2.20], $\mathrm{Gpd}_{S \otimes T}(A \otimes_k B) \geq n+m$, as desired. □

Proof of Theorem 2.4. By Lemma 2.1, there is a finitely generated S-module A (resp. T-module B) such that $\mathrm{Ggldim}\,(S) = \mathrm{Gpd}_S(A)$ and $\mathrm{Ggldim}\,(T) = \mathrm{Gpd}_T(B)$. Then, from Lemma 2.5, we have $\mathrm{Ggldim}\,(S \otimes T) \geq \mathrm{Gpd}_{S \otimes T}(A \otimes B) \geq \mathrm{Gpd}_S(A) + \mathrm{Gpd}_T(B) = \mathrm{Ggldim}\,(S) + \mathrm{Ggldim}\,(T)$. □

Remark 2.6. The inequality in Theorem 2.4 is not definitely. A simple example which give the equality is by consider a quasi-Frobenius ring k (in particular a field) and n and m two arbitrary positive integer and we consider the k-algebras $S := k[x_1, ..., x_n]$ and $T := k[y_1, ..., y_m]$ of polynomials in the variables $x_1, ..., x_n$ and $y_1, ..., y_m$ respectively. We have:

$$S \otimes_k T = k[x_1, ..., x_n] \otimes_k k[y_1, ..., y_m] \cong k[x_1, ..., x_n, y_1, ..., y_m].$$

Thus, by [3, Proposition 2.6] and [4, Theorem 2.1], $\mathrm{Ggldim}\,(S \otimes T) = n + m = \mathrm{Ggldim}\,(S) + \mathrm{Ggldim}\,(T)$.

Let S be a k-algebra. A two-sided S-module A may be regarded as a left module over the k-algebra $S \otimes S^*$ (where S^* is the opposite algebra of S), by setting

$$(s \otimes z^*)a := saz.$$

The k-algebra $S \otimes S^*$ is called the enveloping algebra of S and it is denoted by S^e (for more details see [6]). For a commutative algebra S (and it is the case in this papers) S^* may be identified to S (see [6]). *Note that in the following proofs we still use the notation S^* despite the commutativity of our rings in order to indicate the non-commutative case.* Now let S and T two k algebra. The direct sum $S \oplus T$ with multiplication and operators defined by

$$(s,t)(s',t') := (ss',tt'), \quad k(s,t) := (ks,kt)$$

is then again a k-algebra Σ, called the direct product of S and T. If e_S and e_T are the unit elements of S and T, then (e_S, e_T) is the unit element of Σ.

Proposition 2.7. *Let S and T be two algebras over a ring k and note $\Sigma = S \oplus T$ the direct product of S and T. If a two-sided S-module A is Gorenstein projective S^e-module then, it is a Gorenstein projective Σ^e-module.*

Proof. Consider a complete projective S^e-resolution

$$\mathbf{P} = \cdots \to P_{-1} \xrightarrow{d_{-1}} P_0 \xrightarrow{d_0} P_1 \xrightarrow{d_1} P_2 \xrightarrow{d_2} \cdots .$$

such that $A = Im(d_0)$. Note that , for each $i \in Z$, the S^e-module P_i may be regarded as a two sided S-module by setting

$$sp := (s \otimes 1)p, \quad ps := (1 \otimes s^*)p$$

This second S-module may be also regraded as an S^e-module by setting

$$(s \otimes z^*) * p := spz = (s \otimes 1)(1 \otimes z^*)p = (s \otimes z^*)p$$

Hence, the new S^e-modulation of P_i coincide with the first one. Thus by [6, IX, Proposition 5.2], for each i, P_i is also Σ^e-projective. Moreover, the sequence $\mathbf{P}$ is exact as a Σ^e-module. Indeed, for every $s + t \in \Sigma = S \oplus T$ and $a \in P_i$, we have $d_i((s+t)a) = d_i(s.a) = s.d_i(a) = (s+t)d_i(a)$ (see [6, page 172] for more details about the modulation of Σ^e over the S^e-modules). On the other hand, for every projective Σ^e-module Q we have the identification

$$Hom_{\Sigma^e}(\mathbf{P}, Q) = Hom_{S^e}(\mathbf{P}, Q \otimes_{\Sigma^e} S^e)$$

(see [6, page 173]). Then, $Hom_{\Sigma^e}(\mathbf{P}, Q)$ is exact since $Hom_{S^e}(\mathbf{P}, Q \otimes_{\Sigma^e} S^e)$ is exact (note that $Q \otimes_{\Sigma^e} S^e$ is a projective S^e-module and recall that $\mathbf{P}$ is a complete projective S^e-resolution). □

Proposition 2.8. *Let S and T be two K-algebras and note $\Sigma = S \oplus T$. Then, $Gpd_{\Sigma^e}(\Sigma) \leq max(Gpd_{S^e}(S), Gpd_{T^e}(T))$ with equality if $max(Gpd_{S^e}(S), Gpd_{T^e}(T))$ is finite.*

Proof. If $max(\mathrm{Gpd}_{\Lambda^e}(\Lambda), \mathrm{Gpd}_{\Gamma^e}(\Gamma)) = \infty$ the result is obvious. Otherwise, set $\mathrm{Gpd}_{S^e}(S) = n$ and $\mathrm{Gpd}_{T^e}(T) = m$ and suppose that $n \leq m$. Choose

$$0 \to G_n \to .... \to G_0 \to S \to 0$$

and

$$0 \to G'_m \to .. \to G'_n \to .... \to G'_0 \to T \to 0$$

a Gorenstein projective resolutions of S (as a S^e-module) and T (as a T^e-module) respectively. Then, by Proposition 2.7, they are also a Gorenstein projective resolutions of S and T as Σ^e-modules. Thus, the resolution sum

$$0 \to G'_m \to \cdots \to G_{n+1} \to G_n \oplus G'_n \to \cdots \to G_0 \oplus G'_0 \to S \oplus T \to 0$$

is a Gorenstein projective resolution of $\Sigma = S \oplus T$ as Σ^e-module. Then, $\mathrm{Gpd}_{\Sigma^e}(\Sigma) \leq m$. Now, set $\mathrm{Gpd}_{\Sigma^e}(\Sigma) = l$ (which is finite). Let Q be a projective S^e-module (then it is a projective Σ^e-module by [6, IX, Proposition 5.2] and with the same reason that used in the proof of Proposition 2.7). We have, by [15, Theorem 2.20] and [6, IX, Corollary 5.4], $0 = \mathrm{Ext}^k_{\Sigma^e}(\Sigma, Q) \cong \mathrm{Ext}^k_{S^e}(S, Q)$ for each $k > l$. Thus, by [15, Theorem 2.20] again, $\mathrm{Gpd}_{S^e}(S) \leq l$. Similarly, we have $\mathrm{Gpd}_{T^e}(T) \leq l$. Consequently, we have the desired equality. $\square$

Proposition 2.9. *Let S and T be two algebras over a field k such that they are finitely generated k-modules and $Gpd_{S^e}(S)$ and $Gpd_{T^e}(T)$ are finite. Then,*

$$Gpd_{(S\otimes T)^e}(S \otimes T) \geq Gpd_{S^e}(S) + Gpd_{T^e}(T)$$

.

Proof. Since k is a field and S and T are finitely generated, then S^e and T^e are also finitely generated k-modules and therefore are Noetherian. Thus, S can be regarded as a finitely generated S^e-module. Similarly T is a finitely generated T^e-module. Set $\mathrm{Gpd}_{S^e}(S) = n$ and $\mathrm{Gpd}_{T^e}(T) = m$. Let P (resp. Q) be a projective S^e-module (resp. T^e-module) such that $\mathrm{Ext}^n_{S^e}(S, P) \neq 0$ and $\mathrm{Ext}^m_{T^e}(T, Q) \neq 0$ (by [15, Theorem 2.20]). Thus, $\mathrm{Ext}^n_{S^e}(S, P) \otimes \mathrm{Ext}^m_{S^e}(S, Q) \neq 0$ since k is a field and so, $\mathrm{Ext}^{n+m}_{S^e \otimes T^e}(S \otimes T, P \otimes_k Q) \neq 0$ (by [6, XI, THeorem 3.1]). Seeing that $(S \otimes T)^e \cong S^e \otimes T^e$, we have $\mathrm{Ext}^{n+m}_{(S\otimes T)^e}(S \otimes T, P \otimes_k Q) \neq 0$. Consequently, by [15, Theorem 2.20], $\mathrm{Gpd}_{(S\otimes T)^e}(S \otimes T) \geq n + m$, as desired. $\square$

Now we compare $\mathrm{Gpd}_{S^e}(S)$ with the various other Gorenstein dimensions, namely: $\mathrm{Ggldim}(S)$ and $\mathrm{Ggldim}(S^e)$.

Proposition 2.10. *Let k be a semisimple ring. For any k-algebras S such that $Ggldim(S)$ and $Ggldim(S^e)$ are finite we have $Ggldim(S) \leq Gpd_{S^e}(S)$.*

Proof. Let I be an injective S-module and P an arbitrary projective S-module. Note that since S is a commutative ring, S and S^* may be identified. So, P may be regarded as a projective S^*-module. Moreover, $\mathrm{Ggldim}(S^*)$ is also finite. Then, by [3, Corollary 2.7], $\mathrm{pd}_S(I)$ and $\mathrm{id}_{S^*}(P)$ are finite. Then, by [6, IX, corollary 2.7a], $\mathrm{id}_{S^e}(\mathrm{Hom}_k(I, P))$ is finite. Therefore, $\mathrm{pd}_{S^e}(\mathrm{Hom}_k(I, P))$ is finite (by [3, Corollary 2.7]). Set $\mathrm{Gpd}_{S^e}(S) = n$. Then, for each $i > n$, $\mathrm{Ext}^i_{S^e}(S, \mathrm{Hom}_k(I, P)) = 0$ ([15, Theorem 2.20]). On the other hand, from [6, IX, Corollary 4.4] and since k is semisimple, we have $0 = \mathrm{Ext}^i_{S^e}(S, \mathrm{Hom}_k(I, P)) \cong \mathrm{Ext}^i_S(I, P)$. Hence, $\mathrm{Gpd}_S(I) \leq n$ ([15, Theorem 2.20]). Consequently, by [15, Proposition 2.27], $\mathrm{pd}_S(I) \leq n$. Thus, by Lemma 2.1, $\mathrm{Ggldim}(S) \leq n$. $\square$

Proposition 2.11. *Let S be a quasi-Frobenius algebras over a field k such that $Ggldim(S^e)$ is finite. Then, $Ggldim(S^e) = Gpd_{S^e}(S)$.*

Proof. Let I be an injective S^e-module and P a projective S^e-module. Then, since $\mathrm{Ggldim}(S^e)$ is finite, $\mathrm{pd}_{S^e}(I)$ and $\mathrm{id}_{S^e}(P)$ are finite ([2, Corollary 2.7]). Hence, let X (resp. Y) be a finite S^e-projective (resp. injective) resolution of I (resp. P). Note that X is also a projective resolution of I as an S^*-module since every projective S^e-module is a projective S^*-module (by [6, IX, Corollary 2.4] and since $(S^*)^*) = S$). Then, $\mathrm{pd}_{S^*}(I)$ is finite. So, using [2, Propsition 2.6 and Corollary 2.7], I is a projective S^*-module (recall that S is quasi-Frobenius and so is S^*). Thus $\mathrm{Ext}^n_S(I, P) = 0$ for each $i > 0$. Consequently, $Hom_S(X, Y)$ is a finite S^e-injective resolution of $\mathrm{Hom}_S(I, P)$ (by [6, IX, Proposition 2.6a]). Therefore $\mathrm{id}_{S^e}(\mathrm{Hom}_S(I, P))$ is finite. So, from [3, Corollary 2.7], $\mathrm{pd}_{S^e}(\mathrm{Hom}_S(I, P))$ is finite. Set $n = \mathrm{Gpd}_{S^e}(S)$. Then, $\mathrm{Ext}^i_{S^e}(S, \mathrm{Hom}_S(I, P)) = 0$ for each $i > n$. And from [6, IX, Proposition 2.8a] we have $0 = \mathrm{Ext}^i_{S^e}(S, \mathrm{Hom}_S(I, P)) = \mathrm{Ext}^i_{S^e}(I, P)$ since $\mathrm{Tor}^i_S(I, P) = \mathrm{Ext}^i_S(I, P) = 0$ by the fact that I is a projective S-module. Thus, $\mathrm{Gpd}_{S^e}(I) \leq n$ ([15, Theorem 2.20]). Then, $\mathrm{pd}_{S^e}(I) \leq n$ ([15, Theorem 2.27]). Therefore, by Lemma 2.1, $\mathrm{Ggldim}(S^e) \leq n$. The other inequality follows from the definition of the Gorenstein global dimension.

Corollary 2.12. *Let S be an algebra over a field k such that $Ggldim(S)$ and $Ggldim(S^e)$ are finite. Then, S is a Gorenstein projective S^e-module if and only if, S^e is a quasi-Frobenius.*

Proof. If S is a Gorenstein projective S^e-module, then by Proposition 2.10 and [3, Proposition 2.6], S is a quasi-Frobenius. Thus, by Proposition 2.11, S^e is a quasi-Frobenius. Conversely, if S^e is quasi-Frobenius, then S is a Gorenstein projective S^e-module [3, Proposition 2.6]. □

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[1] Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S.M. Ben Abdellah Fez, Morocco.
E-mail address: mahdou@hotmail.com

[2] Department of Mathematics, Faculty of Science, Box 11201 Zitoune, University Moulay Ismail Meknes, Morocco.
E-mail address: tamekkante@yahoo.fr