

RINGS IN WHICH EVERY HOMOMORPHIC IMAGE SATISFY (STRONG) PROPERTY(A)

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ABSTRACT. The notions of (A) -rings and strong (A) -rings are defined in [6, 12]. In this paper, we study a class of rings in which every homomorphic image satisfy (strong) Property(A).

1. INTRODUCTION

All rings considered in this paper are assumed to be commutative with identity elements and all modules are unitary.

One of important properties of commutative Noetherian rings is that the annihilator of an ideal I consisting entirely of zero-divisors is nonzero [9, p. 56]. However, this result fails for some non-Noetherian rings, even if the ideal I is finitely generated [9, p. 63]. Huckaba and Keller [8] introduced the following: a commutative ring R has property (A) if every finitely generated ideal of R consisting entirely of zero divisors has a non zero annihilator. Property (A) was originally studied by Quentel [13]. Quentel used the term condition (C) for property (A) . The class of commutative rings with property (A) is quite large. For example, Noetherian rings [9, p. 56], rings whose prime ideals are maximal, the polynomial ring $R[X]$ for every ring R and rings whose total ring of quotients are Von Neumann regular. For instance, see [3, 4, 5, 10, 11]. The authors in [12] introduced the following: A ring R is called satisfying strong property (A) if every finitely generated ideal of R which is generated by a finite number of zero-divisors elements of R , has a non zero annihilator.

Let A be a ring and E an A -module. The trivial ring extension of A by E (also called the idealization of E over A) is the ring $R := A \ltimes E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') := (aa', ae' + a'e)$. For the reader's convenience, recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J := I \ltimes E'$ is an ideal of R . However, prime (resp. maximal) ideals of R have the form $p \ltimes E$, where p is a prime (resp., maximal) ideal of A . Homogenous ideals of $R \ltimes E$ have the form $I \ltimes N$ where I is an ideal

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of R , N a submodule of E and $IE \subseteq N$. These ideals play a special role in studying properties of $R \propto E$ and showing how these properties are related to those of R and E . It is shown that a principal ideal $(R \propto E)(a, m)$ is homogeneous if and only if $(R \propto E)(a, m) = Ra \propto (Rm + aM) = (R \propto E)(a, 0) + (R \propto E)(0, m)$, and every ideal of $R \propto E$ is homogeneous if and only if every principal ideal of $R \propto E$ is homogeneous (see [2, Theorem 3.3.]). See for instance [1, 2, 7].

In this paper, we are investigate a particular class of rings satisfying property (A) that we call a fidel (A)-ring (resp., strong fidel (A)-ring). A ring R is called a fidel (A)-ring (resp., strong fidel (A)-ring) if R/I is an (A)-ring (resp., strong (A)-ring) for every ideal I of R . It is clear that any fidel (A)-ring (resp., strong fidel (A)-ring) is an (A)-ring (resp., strong (A)-ring). Proposition 2.3 shows that the class of fidel (A)-rings include a ring with nil krull dimension and Noetherian ring. In Proposition 2.4, we prove that the class of fidel (A)-rings is closed under direct product. Using these results, we construct an example of fidel (A)-rings which does not strong fidel (A)-rings. We also determine in Theorem 2.10 when $S^{-1}R$ is a (resp., strong) fidel (A)-ring for a multiplicatively closed subset S of R consisting of regular elements. Finally, in Proposition 2.14, we gives some conditions under which a ring $R \propto M$ is a (resp., strong) fidel (A)-ring.

2. MAIN RESULTS

In general the homomorphic image of (resp., strong) (A)-ring is not necessary an (resp., strong) (A)-ring (by using [12, Examples 2.6 and 2.7]). In what follows, we introduce the notions of (resp., strong) fidel (A)-rings as:

Definition 2.1. A ring R is called a (resp., strong) fidel (A)-ring if R/I is an (resp., strong) (A)-ring for any ideal I of R .

It is clear that every fidel (A)-ring (resp., strong fidel (A)-ring) is an (A)-ring (resp., strong (A)-ring). The next Example shows that the inverse implications are not true.

Example 2.2 (Example 2.6, [12]). Let R be a ring which is not an (A)-ring, $D := R[X]$ be the polynomial ring over R and for an integer $n \geq 1$ we set $I := (X^n)$ an ideal of D . Then:

- (1) D is a strong (A)-ring. In particular, it is an (A)-ring.
- (2) D is not a (resp., strong) fidel (A)-ring since $D/I = R[X]/(X^n)$ is not an (resp., strong) (A)-ring.

The following proposition prove that the class of fidel (A)-rings includes the class of rings with nil krull dimension. Moreover, every Noetherian ring is a strong fidel (A)-ring.

Proposition 2.3. *Let R be a ring.*

- (1) *If $\dim(R) = 0$, then R is a fidel (A)-ring.*

- (2) If R is Noetherian, then R is a (resp., strong) fidel (A)-ring.
- (3) Let I be an ideal of R . If R is a (resp., strong) fidel (A)-ring, then so is R/I .

Proof. (1) If $\dim(R) = 0$, then for each ideal I of R we have $\dim(R/I) = 0$. So, every prime ideal of R/I is maximal. Hence, by [7, Corollary 2.12], R/I is an (A)-ring. Consequently, R is a fidel (A)-ring.

(2) From [9, p. 56], over Noetherian ring, the annihilator of an ideal consisting entirely of zero-divisors is nonzero. Hence, every Noetherian ring is a (resp., strong) fidel (A)-ring. On the other hand, if R is Noetherian, then so is R/I for any ideal I of R . Therefore, R/I is a (resp., strong) (A)-ring. Hence, R is a (resp., strong) fidel (A)-ring.

(3) Let R be a (resp., strong) fidel (A)-ring. Then, R/I is an (resp., strong) (A)-ring for any ideal I of R . Thus, $(R/I)/(J/I) \simeq R/J$ is an (resp., strong) (A)-ring for any ideal J of R that contain I . Therefore, R/I is a (resp., strong) fidel (A)-ring. $\square$

As with the class of (A)-ring ([6, Proposition 1.3]), the class of fidel (A)-rings is closed under direct product.

Proposition 2.4. *A direct product of rings $R = \prod_{i \in J} R_i$ (for some indexing set J) is a fidel (A)-ring if and only if R_i is a fidel (A)-ring for each $i \in J$.*

Proof. Suppose that R is a fidel (A)-ring. Then, for each an arbitrary ideal I_i of R_i , $\prod_{i \in J} R_i / \prod_{i \in J} I_i \cong \prod_{i \in J} R_i / I_i$ is an (A)-ring. Hence, by [6, Proposition 1.3], R_i / I_i is an (A)-ring for each $i \in J$. Consequently, R_i is a fidel (A)-ring for each $i \in J$.

Conversely, let K be an ideal of R . It can be written as $K = \prod_{i \in J} I_i$ where I_i is an ideal of R_i for each $i \in J$. By [6, Proposition 1.3], $R/K \cong \prod_{i \in J} R_i / I_i$ is an (A)-ring since R_i is a fidel (A)-ring for each $i \in J$. Consequently, R is a fidel (A)-ring. $\square$

The bellow proposition allows us to give an example of a fidel (A)-ring which is not strong fidel (A)-ring.

Example 2.5. Let $D = K[X]$ be the polynomial ring over a field K and set $D \times D$ the direct product of D by D . Then:

- (1) D is a (resp., strong) fidel (A)-ring (see that D is Noetherian).
- (2) $D \times D$ is a fidel (A)-ring by Proposition 2.4.
- (3) $D \times D$ is not a strong (A)-ring by [12, Example 2.1]. In particular, it is not a strong fidel (A)-ring.

Remark 2.6. Example 2.5 shows that the class of strong fidel (A)-rings is not closed under direct product.

A ring R is called a Von Neumann regular if for every $a \in R$ there exists an $x \in R$ such that $a = axa$. A ring is Von Neumann regular if and only if every finitely generated ideal of R is generated by an idempotent.

Proposition 2.7. *Every Von Neumann regular ring is a strong fidel (A)-ring.*

Proof. Consider I an ideal of a ring R . Let $J = \sum_{i=1}^n \bar{a}_i R/I$ be an ideal of R/I . If R is Von Neumann regular, then the ideal $I = \sum_{i=1}^n a_i R$ is generated by an idempotent e . Hence, $J = \bar{e}R/I$. So, $(1 - \bar{e})J = 0$. Consequently, R is a strong fidel (A)-ring.

Remark 2.8. Example 2.5 shows that strong fidel (A)-ring is not necessary Von Neumann regular (see that a polynomial ring is never Von Neumann regular).

Proposition 2.9. *Let R be a coherent fidel (A)-ring and I is a finitely generated ideal of R then, the total quotient ring $T(R/I)$ is Von Neumann regular in the following cases:*

- (1) $I = \text{Nil}(R)$.
- (2) I is a prime ideal of R .

Proof. The ring R/I is coherent since I is finitely generated and R is coherent. Moreover, R/I is reduced in the two cases. Therefore, R/I is a reduced coherent (A)-ring. Consequently, by [8, Theorem 3], $T(R/I)$ is Von Neumann regular.

Theorem 2.10. *Let R be a ring and let S a multiplicatively closed subset of R consisting of regular elements. If every ideal I of R such that $I \cap S = \emptyset$ satisfies the following property:*

$$\forall s \in S, \forall i \notin I \Rightarrow is \notin I.$$

Then, R is a (resp., strong) fidel (A)-ring if and only if $S^{-1}R$ is a (resp., strong) fidel (A)-ring.

Before proving this Theorem, we establish and recall the following Lemmas.

Lemma 2.11. *Let R be a ring. Let S a multiplicatively closed subset of R consisting of regular elements. For an ideal I of R we set $p_I : R \rightarrow R/I$ the canonical projection and $\bar{S} = p(S)$. Consider $i_S : R \rightarrow S^{-1}R$ the ring homomorphism defined by $i_S(a) = a/1$. Then:*

- (1) $\bar{S}^{-1}(R/I) \simeq S^{-1}R/S^{-1}I$.
- (2) $J = S^{-1}(i_S^{-1}(J))$ for any ideal J of $S^{-1}R$.

Proof. (1) Consider $\alpha : S^{-1}R \rightarrow \bar{S}^{-1}(R/I)$ defined by $r/s \mapsto \bar{r}/\bar{s}$. It is easy to verify that α is a surjective ring homomorphism and its kernel coincides with $S^{-1}I$. Therefore, $\bar{S}^{-1}(R/I) \simeq S^{-1}R/S^{-1}I$.

(2) Let J be any ideal of $S^{-1}R$. Set $I = i_S^{-1}(J)$. Clearly, I is an ideal of R . We have to prove that $J = S^{-1}I$. Let $\frac{a}{s} \in J$. Thus, $\frac{s}{1} \cdot \frac{a}{s} = \frac{a}{1} \in J$. So, $a \in I$. Hence, $J \subseteq S^{-1}I$. The other containment is clear as $i_S(I) \subseteq J$. Therefore, $J = S^{-1}I = S^{-1}(i_S^{-1}(J))$. $\square$

Lemma 2.12 (Proposition 2.14, [6] and Lemma 2.3, [12]). *Let R be a ring and S a multiplicatively closed subset of R consisting of regular elements. Then R is an (resp., strong) (A)-ring if and only if so is $S^{-1}R$.*

Proof of Theorem 2.10. We use the notation of Lemma 2.11.

Assume that R is a (resp., strong) fidel (A) -ring, then R/I is an (resp., strong) (A) -ring for any ideal I of R . Let J be any ideal of $S^{-1}R$. Then, $J = (J \cap R).S^{-1}R = I.S^{-1}R$ with $I = J \cap R$. Two cases are then possible:

Case 1: $I \cap S = J \cap R \cap S = J \cap S \neq \emptyset$. In this case $J = I.S^{-1}R = S^{-1}R$. Then, $S^{-1}R/J$ is an (resp., strong) A -ring.

Case 2: $I \cap S = \emptyset$. In this case $\bar{S}$ is a multiplicatively closed subset of R/I consisting of regular elements. Indeed, let $s \in S$ and $r \in R$ such that $\bar{s}.\bar{r} = \bar{0}$. Then, $s.r \in I$. By hypothesis, $r \in I$. Therefore, by lemmas 2.11 and 2.12, $(S^{-1}R)/J = S^{-1}R/I.S^{-1}R \simeq \bar{S}^{-1}(R/I)$ is an (resp., strong) (A) -ring. Consequently, $S^{-1}R$ is a (resp., strong) fidel (A) -ring.

Conversely, assume that $S^{-1}R$ is a (resp., strong) fidel (A) -ring. Then, $S^{-1}R/S^{-1}I \simeq \bar{S}^{-1}(R/I)$ is an (resp., strong) (A) -ring for any ideal I of R . Therefore, R/I is an (resp., strong) (A) -ring for any ideal I of R by lemma 2.12. Consequently, R is a (resp., strong) fidel (A) -ring. $\square$

As a consequence of the bellow theorem, we give the following Corollary:

Corollary 2.13. *Let D be an integral domain, E be a D -module and let $R := D \rtimes E$ be the trivial ring extension of D by E . Then R is a fidel (A) -ring.*

Proof. Let E be a D -module and set $R := D \rtimes E$. Let $S = D \setminus \{0\}$ which is a multiplicatively closed subset of R consisting of regular elements. Then R is a fidel (A) -ring if and only if so is $S^{-1}R(= S^{-1}D \rtimes S^{-1}E)$ by Theorem 2.10 since for every ideal I of R such that $I \cap S = \emptyset$ satisfies the following property: $\forall s \in S, \forall i \notin I \Rightarrow is \notin I$. On the other hand, $K := S^{-1}D$ is a fidel (A) -ring (since D is an integral domain). The only proper ideal of $K \rtimes S^{-1}E$ is $0 \rtimes S^{-1}E$. Then the possible proper homomorphic images of $K \rtimes S^{-1}E$ is isomorphic to K (which is an (A) -ring). Therefore, $K \rtimes S^{-1}E$ is a fidel (A) -ring. Consequently, R is a fidel (A) -ring.

For a fidel (A) -ring R and a R -module M , the trivial extension $R \rtimes M$ of R by M is not fidel (A) -ring in general (see [12, Example 2.7]).

The next result gives some conditions under which a ring $R \rtimes M$ is a (resp., strong) fidel (A) -ring.

Proposition 2.14. *Let R be a ring and M an R -module such that every ideal of $R \rtimes M$ is homogenous.*

- (1) *Let M be flat and the homomorphic image of M is flat. If $R \rtimes M$ is a fidel (A) -ring, then so is R . The converse is true if M is finitely generated.*
- (2) *If M is free R -module and the homomorphic image of M is free, then $R \rtimes M$ is a fidel strong (A) -ring if and only if so is R .*

Proof. Suppose $R \rtimes M$ is a (resp., strong) fidel (A) -ring. Then $R \rtimes M/I \rtimes N \simeq R/I \rtimes M/N$ is an (resp., strong) (A) -ring for any homogenous ideal $I \rtimes N$ of $R \rtimes M$. Since M/N is flat (resp., free) R -module, it follows by [1, Theorem 16.]

(resp., [12, Theorem 2.1]) that R/I is an (resp., strong) (A) -ring for any ideal I of R . Therefore, R is a (resp., strong) fidel (A) -ring. Conversely, Suppose R is a (resp., strong) fidel (A) -ring. Suppose M be finitely generated flat R -module (resp., free R -module) such that the homomorphic image of M is flat (resp., free). Then, M/N is finitely generated flat R/I -module (resp., free R/I -module). It follows by [1, Theorem. 16](resp., [12, Theorem 2.1]), that $R/I \propto M/N$ is an (resp., strong) (A) -ring for any ideal $I \propto N$ of $R \propto M$. Therefore, $R \propto M$ is a (resp., strong) fidel (A) -ring.

We round off this paper by an other example of fidel A -ring which is not strong fidel A -ring.

Example 2.15. Let K be a field and let $A = K[[X]]$ be the power series ring over K . Let $J = (X)$ be the ideal of A generated by X . Consider $A \bowtie J$ be the amalgamation of A along J . Then:

- (1) A is a fidel strong (A) -ring. In particular, it is a fidel (A) -ring.
- (2) $A \bowtie J$ is not a strong (A) -ring. In particular, it is not a fidel strong (A) -ring.

Proof. (1) Clear since A is an integral domain.

(2) By [12, Example 3.3].

REFERENCES

1. M. M. Ali; *Idealization and theorems of D. D. Anderson II*, Comm. Algebra. **35** (2007), 2767-2792.
2. D. D. Anderson, M. Winders; *Idealization of a module*, Comm. Algebra, No. 1, Vol. 1 Spring(2009).
3. F. Azarpanah, O. A. S. Karamzadeh and A. Rezai Aliabad; *On ideals consisting entirely of zero divisors*, Comm. Algebra. **28** (2000), 1061-1073.
4. M. Henriksen and M. Jerison; *The space of minimal prime ideals of a commutative ring*, Trans. Amer. Math. Soc. (115) (1965), 110-130.
5. G. Hinkle and J.A. Huckaba; *The generalized Kronecker function ring and the ring $R(X)$* , J. Reine Angew. Math. 292 (1977), 25-36.
6. C. Y. Hong, N. K. Kim, Y. Lee and S. J. Ryu; *Rings with Property (A) and their extensions*, J. Algebra. (315) (2007), 612-628.
7. J. A. Huckaba; *Commutative rings with zero divisors*, Marcel Dekker Inc, New York, (1988).
8. J. A. Huckaba and J. M. Keller; *Annihilation of ideals in commutative rings*, Pacific J. Math. (83) (1979), 375-379.
9. I. Kaplansky; *Commutative rings*, Allyn and Bacon, Boston, 1970.
10. T. G. Lucas; *Two annihilator conditions: Property (A) and $(a.c)$* , Comm. Algebra (14) (1986), 557-580.
11. T. G. Lucas; *The diameter of a zero divisors graph*, J. Algebra (301) (2006), 174-193.
12. N. Mahdou and A. Rahmouni. Hassani; *On strong (A) -Rings*, Mediterr. J. Math, No. 2, Vol. 9 (2012), 395-405.
13. Y. Quentel; *Sur la compacit du spectre minimal d'un anneau*, Bull. Soc. Math, France (99) (1971), 265-272.

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