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MOTION PROPERTIES OF THE VARIABLE MASS SMALLEST BODY IN CYCLIC KITE CONFIGURATION WITH KERR-LIKE OBLATE HETEROGENEOUS PRIMARIES

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Abstract. This paper presents the motion of the smallest body in the 5-body problem where 4 bodies construct the cyclic kite configuration. Out of five bodies, four bodies are situated at the vertices of a kite and the shapes of these bodies are considered as oblate heterogeneous by the supposition that these bodies are spinning around their own axes. It is also supposed that the smallest fifth body is varying its mass, and its motion is affected by the coriolis as well as centrifugal forces. We numerically illustrate the locations of fixed points, their stability, the zero-velocity surfaces with projections and Poincaré surfaces of section by the determined equations of motion.

1. Introduction

The restricted problem can have various types of configuration with various perturbations. These various types of configurations and various types of perturbations in the restricted problems are studied by many researchers. Analysis of restricted problems are made by many ways like assuming by number of bodies, the configurations of the bodies, shapes of the bodies, the movement of the bodies and the external forces working on these bodies etc. Therefore, this problem is application based problem. And hence the study of this problem is directly used by the space agencies in the world-wide.

This types of restricted problems are investigated by many scientists, some of them are as follow: [17], [15], [25], [26], [2], [14], [27], [23], [20], [12, 13], [29], [28], [7], [21], [11].

In our universe all the celestial bodies varies their masses therefore we can not study the motion properties by ordinary method hence we will use Jeans law and Meshcherskii space transformations. By using above said transformations, various researchers have investigated the mass variation effects as: [22], [30], [10, 8, 9], [16], [5, 6], [1].

The arrangement of the manuscript is as follows: The introduction of the problem is presented in section 1. The description of the problem with equations of motion are given in section 2. Section 3 represents the locations of fixed points

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while their stability states are given in section 4. Sections 5 performed the zero-velocity surfaces with projections while PSS are illustrated in sections 6. The paper completes with conclusion in section 7.

2. Description of the problem with equations of motion

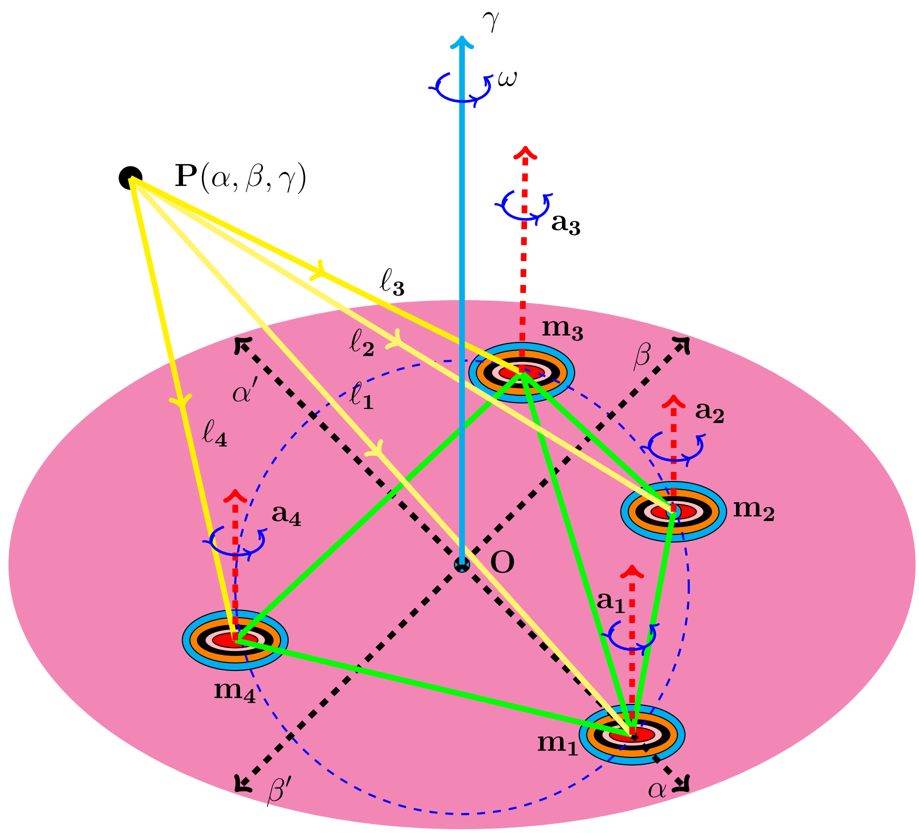


Figure 1. Cyclic kite configuration with kerr-like oblate heterogeneous primaries

Let the masses of four oblate heterogeneous bodies be m_i (i = 1, 2, 3, 4), which are rotating about their own axes with angular velocities a_i respectively. These bodies (equal masses, equal heterogeneous parameters $(\rho_{i,1})$ and equal angular velocities) are situated at the vertices of a cyclic kite (the radius of circle is d). The formation of kite is assumed to be in two parts: first part of the kite is formed by $m_1m_3m_4$ which is an equilateral triangle with side ℓ, and second part of the kite is formed by $m_1m_2m_3$ which is an isosceles triangle with two equal sides $m_1m_2 = m_2m_3 = d$. In this way the cyclic kite configuration is obtained with the relation $\ell = \sqrt{3}\,d$. Hence the coordinates of vertices of kite m_1, m_2, m_3 and m_4 will be $(d, 0, 0)$, $(\frac{d}{2}, \frac{\sqrt{3}\,d}{2}, 0)$, $(-\frac{d}{2}, \frac{\sqrt{3}\,d}{2}, 0)$ and $(-\frac{d}{2}, -\frac{\sqrt{3}\,d}{2}, 0)$ respectively (the complete view is given in figure (1)).

Under the gravitational forces of these four kerr-like primaries, the transition from Newtonian to beyond Newtonian with transition parameter $\epsilon \in [0, 1]$ and the effects of the coriolis as well as centrifugal forces with parameters ϵ_1 as well as ϵ_2 respectively, we will discuss here the motion properties of the smallest fifth body which is varying its mass.

Using the procedure given by [3] and [18], one can write the potential between two kerr-like oblate heterogeneous bodies moving in the same plane as

$$\begin{aligned}\Omega_{i\,j} &= -\frac{G\,m_i\,m_j}{r_{i\,j}} - \frac{G\,(m_j\,\rho_{i\,1}\,+\,m_i\,\rho_{j\,1})}{2\,r_{i\,j}^3} + \frac{\epsilon\,m_i\,m_j}{r_{i\,j}^2}(a_i\,Cos\,\theta_i\,+\,a_j\,Cos\,\theta_j) \\ &\quad + \frac{\epsilon^2\,m_i\,m_j}{2\,r_{i\,j}^3}\left\{m_j^2\,+\,a_i^2(3\,Cos^2\,\theta_i\,-\,1) + \,a_j^2\,(3\,Cos^2\,\theta_j\,-\,1)\right\}, \\ &\quad i,\,j\,=\,1,\,2,\,3,\,4,\,\text{provided}\,i \neq j,\end{aligned} \tag{2.1}$$

where m_i and m_j are the masses of the bodies, $r_{i\,j}$ is the distance between both bodies, G is the universal gravitational constant, $\rho_{i\,1}$ and $\rho_{j\,1}$ are the density parameters of the heterogeneous bodies, ϵ is the transition parameter, a_i and a_j are the angular velocities of the bodies, θ_i and θ_j are the inclination of the bodies with respect to the coordinate axes. Following the procedure used by [3] and utilizing Eq. (2.1) by fixing the units as $m_1 + m_2 + m_3 + m_4 = 1$, $a_1 + a_2 + a_3 + a_4 = 1$, $G = 1$ and $d = 1$. If $m_1 = m_2 = m_3 = m_4 = \mu$ and $a_1 = a_2 = a_3 = a_4 = a$ then $\mu = 1/4$ and $a = 1/4$, and also the heterogeneous parameters $\rho_{i\,1} = \rho_{j\,1} = h$, one can write the mean motion ν of the system as

$$\nu^2 = \frac{1}{8}(5+2\sqrt{3}) + \frac{1}{8}(51+8\sqrt{3})\,h - \frac{13}{16}\,\epsilon - \frac{5}{512}(27+4\sqrt{3})\,\epsilon^2. \tag{2.2}$$

One can determine the equations of motion for the variable mass smallest body by utilizing the method used by [4] as

$$\begin{aligned}\ddot{\alpha}\,-\,2\,\nu\,\epsilon_1\,\dot{\beta} \;+\; \frac{\dot{m}}{m}\,(\dot{\alpha}\,-\,\nu\,\epsilon_1\,\beta) &= \frac{\partial\,W}{\partial\,\alpha}, \\ \ddot{\beta}\,+\,2\,\nu\,\epsilon_1\,\dot{\alpha} \;+\; \frac{\dot{m}}{m}\,(\dot{\beta}\,+\,\nu\,\epsilon_1\,x) &= \frac{\partial\,W}{\partial\,\beta}, \\ \ddot{\gamma} \;+\; \frac{\dot{m}}{m}\,\dot{\gamma} &= \frac{\partial\,W}{\partial\,\gamma}.\end{aligned} \tag{2.3}$$

with

$$
\begin{aligned}
W \;=\; & \frac{\nu^2\,\epsilon_2}{2}\,(\alpha^2\,+\,\beta^2)\,+\,\mu\,\left(\frac{1}{\ell_1}\,+\,\frac{1}{\ell_2}+\frac{1}{\ell_3}+\frac{1}{\ell_4}\right)\,+\,\frac{h}{2}\,\left(\frac{1}{\ell_1^3}\,+\,\frac{1}{\ell_2^3}+\frac{1}{\ell_3^3}+\frac{1}{\ell_4^3}\right)\\
& -\frac{3\,h\,\gamma^2}{2}\,\left(\frac{1}{\ell_1^5}\,+\,\frac{1}{\ell_2^5}+\frac{1}{\ell_3^5}+\frac{1}{\ell_4^5}\right)\,-\,\epsilon\,\mu\,a\,\left(\frac{1}{\ell_1^2}\,\mathrm{Cos}\,\phi_1+\frac{1}{\ell_2^2}\,\mathrm{Cos}\,\phi_2\right.\\
& \left.+\frac{1}{\ell_3^2}\,\mathrm{Cos}\,\phi_3\,+\frac{1}{\ell_4^2}\,\mathrm{Cos}\,\phi_4\right)\,-\,\frac{\epsilon^2}{2}\,\left\{\frac{\mu^3}{\ell_1^3}+\frac{\mu\,a^2}{\ell_1^3}(3\,\mathrm{Cos}^2\,\phi_1-1)\,+\,\frac{\mu^3}{\ell_2^3}\right.\\
& +\frac{\mu\,a^2}{\ell_2^3}(3\,\mathrm{Cos}^2\,\phi_2-1)\,+\frac{\mu^3}{\ell_3^3}\,+\frac{\mu\,a^2}{\ell_3^3}(3\,\mathrm{Cos}^2\,\phi_3-1)\,+\,\frac{\mu^3}{\ell_4^3}\\
& \left.+\frac{\mu\,a^2}{\ell_4^3}(3\,\mathrm{Cos}^2\,\phi_4-1)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\ell_1^2 \;&=\; (\alpha\,-\,1\,)^2+\beta^2+\gamma^2,\;\; \ell_2^2\,=\,\left(\alpha\,-\,\frac{1}{2}\right)^2+\left(\beta-\frac{\sqrt{3}}{2}\right)^2+\gamma^2,\\
\ell_3^2 \;&=\; \left(\alpha\,+\,\frac{1}{2}\right)^2+\left(\beta-\frac{\sqrt{3}}{2}\right)^2+\gamma^2,\;\; \ell_4^2\,=\,\left(\alpha\,+\,\frac{1}{2}\right)^2+\left(\beta+\frac{\sqrt{3}}{2}\right)^2+\gamma^2,\\
\phi_1 \;&=\; \tan^{-1}\left(\frac{\beta}{\alpha\,-\,1}\right),\;\; \phi_2\,=\,\tan^{-1}\left(\frac{\beta-\frac{\sqrt{3}}{2}}{\alpha\,-\,\frac{1}{2}}\right),\\
\phi_3 \;&=\; \tan^{-1}\left(\frac{\beta-\frac{\sqrt{3}}{2}}{\alpha\,+\,\frac{1}{2}}\right),\;\; \phi_4\,=\,\tan^{-1}\left(\frac{\beta+\frac{\sqrt{3}}{2}}{\alpha\,+\,\frac{1}{2}}\right).
\end{aligned}
$$

We can not investigate this model by ordinary methods because of variable mass of the infinitesimal body. Therefore, Jean's law [19] and Meshcherskii space-time transformations [24] will be used as:

$$
m \;=\; m_0\,e^{-\delta_1\,t}\;\text{ and }\;(\alpha,\beta,\gamma)\,=\,\delta_2^{-1/2}\,(x,\,y,\,z), \tag{2.4}
$$

where δ_1 is a constant and $\delta_2=\dfrac{m}{m_0}$, where m_0 is the mass of smallest body at time t = 0. And also

$$
\begin{aligned}
\dot\alpha \;&=\; \delta_2^{-1/2}\left(\dot x+\frac{1}{2}\,\delta_1\,x\right),\;\; \ddot\alpha\,=\,\delta_2^{-1/2}\left(\ddot x+\delta_1\,\dot x\,+\,\frac{1}{4}\,\delta_1^2\,x\right),\\
\dot\beta \;&=\; \delta_2^{-1/2}\left(\dot y+\frac{1}{2}\,\delta_1\,y\right),\;\; \ddot\beta\,=\,\delta_2^{-1/2}\left(\ddot y+\delta_1\,\dot y\,+\,\frac{1}{4}\,\delta_1^2\,y\right),\\
\dot\gamma \;&=\; \delta_2^{-1/2}\left(\dot z+\frac{1}{2}\,\delta_1\,z\right),\;\; \ddot\gamma\,=\,\delta_2^{-1/2}\left(\ddot z+\delta_1\,\dot z\,+\,\frac{1}{4}\,\delta_1^2\,z\right).
\end{aligned}
\tag{2.5}
$$

By Eqs. (2.3), (2.4) and (2.5), we obtain

$$\begin{aligned} \ddot{x} - 2\,\nu\,\epsilon_1\,\dot{y} &= \frac{\partial H}{\partial x}, \\ \ddot{y} + 2\,\nu\,\epsilon_1\,\dot{x} &= \frac{\partial H}{\partial y}, \\ \ddot{z} &= \frac{\partial H}{\partial z}, \end{aligned} \tag{2.6}$$

with

$$\begin{aligned} H &= \frac{\nu^2\,\epsilon_2}{2}\,(x^2 + y^2) + \frac{\delta_1^2}{8}\,(x^2 + y^2 + z^2) + \mu\,\delta_2^{3/2}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right) \\ &+ \frac{h\,\delta_2^{5/2}}{2}\left(\frac{1}{r_1^3} + \frac{1}{r_2^3} + \frac{1}{r_3^3} + \frac{1}{r_4^3}\right) - \frac{3\,h\,z^2\,\delta_2^{5/2}}{2}\left(\frac{1}{r_1^5} + \frac{1}{r_2^5} + \frac{1}{r_3^5} + \frac{1}{r_4^5}\right) \\ &- \frac{\epsilon^2\,\delta_2^{5/2}}{2}\left(\frac{\mu^3}{r_1^3} + \frac{\mu\,a^2}{r_1^3}(3\,\mathrm{Cos}^2\,\theta_1 - 1) + \frac{\mu^3}{r_2^3} + \frac{\mu\,a^2}{r_2^3}(3\,\mathrm{Cos}^2\,\theta_2 - 1)\right. \\ &\left. + \frac{\mu^3}{r_3^3} + \frac{\mu\,a^2}{r_3^3}(3\,\mathrm{Cos}^2\,\theta_3 - 1) + \frac{\mu^3}{r_4^3} + \frac{\mu\,a^2}{r_4^3}(3\,\mathrm{Cos}^2\,\theta_4 - 1)\right) \\ &- \epsilon\,\mu\,a\,\delta_2^2\left(\frac{1}{r_1^2}\,\mathrm{Cos}\,\theta_1 + \frac{1}{r_2^2}\,\mathrm{Cos}\,\theta_2 + \frac{1}{r_3^2}\,\mathrm{Cos}\,\theta_3 + \frac{1}{r_4^2}\,\mathrm{Cos}\,\theta_4\right), \\ r_1^2 &= \left(x - \sqrt{\delta_2}\right)^2 + y^2 + z^2, \quad r_2^2 = \left(x - \frac{1}{2}\sqrt{\delta_2}\right)^2 + \left(y - \frac{\sqrt{3}}{2}\sqrt{\delta_2}\right)^2 + z^2, \\ r_3^2 &= \left(x + \frac{1}{2}\sqrt{\delta_2}\right)^2 + \left(y - \frac{\sqrt{3}}{2}\sqrt{\delta_2}\right)^2 + z^2, \\ r_4^2 &= \left(x + \frac{1}{2}\sqrt{\delta_2}\right)^2 + \left(y + \frac{\sqrt{3}}{2}\sqrt{\delta_2}\right)^2 + z^2, \\ \theta_1 &= \tan^{-1}\left(\frac{y}{x - \sqrt{\delta_2}}\right), \quad \theta_2 = \tan^{-1}\left(\frac{y - \frac{\sqrt{3}}{2}\sqrt{\delta_2}}{x - \frac{1}{2}\sqrt{\delta_2}}\right), \\ \theta_3 &= \tan^{-1}\left(\frac{y - \frac{\sqrt{3}}{2}\sqrt{\delta_2}}{x + \frac{1}{2}\sqrt{\delta_2}}\right), \quad \theta_4 = \tan^{-1}\left(\frac{y + \frac{\sqrt{3}}{2}\sqrt{\delta_2}}{x + \frac{1}{2}\sqrt{\delta_2}}\right). \end{aligned}$$

The obtained Eq. (2.6) is the equations of motion for the variable mass smallest body in the perturbed cyclic kite configuration with Kerr-like oblate heterogeneous primaries.

3. Fixed points

The fixed points for this model can be obtained from Eq. (2.6) by putting to the derivatives with respect to time t by zero. And hence $\partial H/\partial x = 0$ and $\partial H/\partial y = 0$.

Now, we solve numerically the partial derivatives for various values of the parameters used to perform the locations of the fixed points. These locations of fixed points are presented in figures (2) in four cases (Case-1: Unperturbed case (figure (2(a))), Case-2: perturbations due to coriolis, heterogeneous and beyond newtonian are taken (figure (2(b))), Case-3: perturbations due to coriolis and variable mass are taken (figure (2(c))) and Case-4: Complete perturbations are taken (figure (2(d)))). Cases- 1, 3, 4 have only five fixed points while case-2 have four more fixed points i.e. nine fixed points exist in this case. From figures (2(a)), (2(b)), (2(c)) and (2(d)), we observed that the fixed points L_2 and L_3 are symmetrical about ordinate while the fixed points L_3 and L_4 are symmetrical about abscissa.

4. Stability

Here we are going to examine the stability of the fixed points for which we need to shift to the fixed points (x_{01}, y_{01}, z_{01}) as

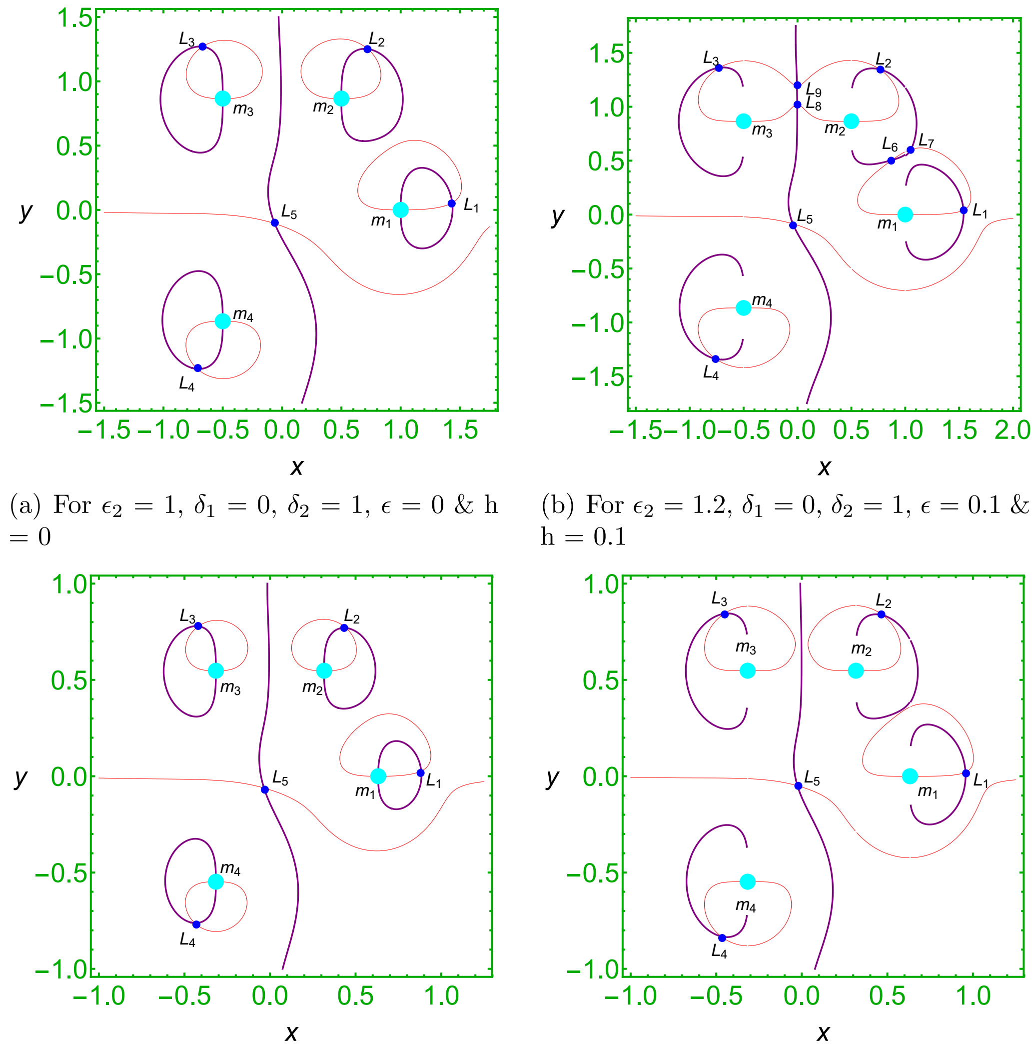
$$\begin{aligned} x &= x_{01} + x_{02}, \\ y &= y_{01} + y_{02}, \\ z &= z_{01} + z_{02}, \end{aligned} \tag{4.1}$$

where (x_{02}, y_{02}, z_{02}) is the slight shift from the fixed points.
Hence the variational equations of the equations of motion (2.6) as:

$$\begin{aligned} &\ddot{x}_{02} - 2\,\nu\,\epsilon_1\,\dot{y}_{02} = H^0_{x\,x}\,x_{02} + H^0_{x\,y}\,y_{02} + H^0_{x\,z}\,z_{02}, \\ &\ddot{y}_{02} + 2\,\nu\,\epsilon_1\,\dot{x}_{02} = H^0_{y\,x}\,x_{02} + H^0_{y\,y}\,y_{02} + H^0_{y\,z}\,z_{02}, \\ &\ddot{z}_{02} = H^0_{z\,x}\,x_{02} + H^0_{z\,y}\,y_{02} + H^0_{z\,z}\,z_{02}. \end{aligned} \tag{4.2}$$

where superscript zero denotes the value of the derivatives of H at fixed point. The subscripts of H in Eq. (4.2) represent the derivatives with respect to respective variables.

The system (4.2) can be written in the phase space as



(a) For $\epsilon_2 = 1$, $\delta_1 = 0$, $\delta_2 = 1$, $\epsilon = 0$ & h = 0

(b) For $\epsilon_2 = 1.2$, $\delta_1 = 0$, $\delta_2 = 1$, $\epsilon = 0.1$ & h = 0.1

(c) For $\epsilon_2 = 1.2$, $\delta_1 = 0.2$, $\delta_2 = 0.4$, $\epsilon = 0$ & h = 0

(d) For $\epsilon_2 = 1.2$, $\delta_1 = 0.2$, $\delta_2 = 0.4$, $\epsilon = 0.1$ & h = 0.1

FIGURE 2. Positions of fixed points in various cases

$$
\begin{aligned}
\dot{x}_{02} &= x_{03}, \\
\dot{y}_{02} &= y_{03}, \\
\dot{z}_{02} &= z_{03}, \\
\dot{x}_{03} &= H^0_{x\,x}\, x_{02} + H^0_{x\,y}\, y_{02} + H^0_{x\,z}\, z_{02} + 2\,\nu\,\epsilon_1\, y_{03}, \\
\dot{y}_{03} &= H^0_{y\,x}\, x_{02} + H^0_{y\,y}\, y_{02} + H^0_{y\,z}\, z_{02} - 2\,\nu\,\epsilon_1\, x_{03}, \\
\dot{z}_{03} &= H^0_{z\,x}\, x_{02} + H^0_{z\,y}\, y_{02} + H^0_{z\,z}\, z_{02}.
\end{aligned}
\tag{4.3}
$$

We will use the Meshcherskii-space-time inverse transformations due to variations in mass. These are as follow:

$$\begin{aligned} x_{04} &= \frac{1}{\sqrt{\delta_2}}\, x_{02}, \quad x_{05} = \frac{1}{\sqrt{\delta_2}}\, x_{03} \\ y_{04} &= \frac{1}{\sqrt{\delta_2}}\, y_{02}, \quad y_{05} = \frac{1}{\sqrt{\delta_2}}\, y_{03}, \\ z_{04} &= \frac{1}{\sqrt{\delta_2}}\, z_{02}, \quad z_{05} = \frac{1}{\sqrt{\delta_2}}\, z_{03}. \end{aligned} \tag{4.4}$$

From Eqs. (4.3) and (4.4), one can write as

$$\dot{\mathbf{X}} = \mathbf{M}\,\mathbf{X}, \tag{4.5}$$

where

$$\dot{\mathbf{X}} = \begin{pmatrix} \dot{x}_{04} \\ \dot{y}_{04} \\ \dot{z}_{04} \\ \dot{x}_{05} \\ \dot{y}_{05} \\ \dot{z}_{05} \end{pmatrix}, \mathbf{M} = \begin{pmatrix} \frac{1}{2}\delta_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2}\delta_1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2}\delta_1 & 0 & 0 & 1 \\ H^0_{x\,x} & H^0_{x\,y} & H^0_{x\,z} & \frac{1}{2}\delta_1 & 2\,\nu\,\epsilon_1 & 0 \\ H^0_{y\,x} & H^0_{y\,y} & H^0_{y\,z} & -2\,\nu\,\epsilon_1 & \frac{1}{2}\delta_1 & 0 \\ H^0_{z\,x} & H^0_{z\,y} & H^0_{z\,z} & 0 & 0 & \frac{1}{2}\delta_1 \end{pmatrix} \,\&\, \mathbf{x} = \begin{pmatrix} x_{04} \\ y_{04} \\ z_{04} \\ x_{05} \\ y_{05} \\ z_{05} \end{pmatrix}. \tag{4.6}$$

Hence the characteristic equation of the matrix M as

$$\lambda^6 + H_5\,\lambda^5 + H_4\,\lambda^4 + H_3\,\lambda^3 + H_2\,\lambda^2 + H_1\,\lambda + H_0, \tag{4.7}$$

where,

$$
\begin{aligned}
H_5 =&\; - 3\,\delta_1\\
H_4 =&\; \frac{15}{4}\,\delta_1^2 + 4\,\epsilon_1^2\,\nu^2 \,-\, (H_{x\,x} + H_{y\,y} + H_{z\,z}),\\
H_3 =&\; -\,\frac{5}{2}\,\delta_1^3 - 8\,\epsilon_1^2\,\delta_1\,\nu^2 \,+\, 2\,\delta_1\,(H_{x\,x} + H_{y\,y} + H_{z\,z}),\\
H_2 =&\; \frac{1}{16}\{15\,\delta_1^4 + 96\,\epsilon_1^2\,\delta_1^2\,\nu^2 \,-24\,\delta_1^2\,(H_{x\,x} \,+\, H_{y\,y} \,+\, H_{z\,z}) - 64\epsilon_1^2\,\nu^2\,H_{z\,z}\\
&\,+\,16(H_{x\,x}\,H_{y\,y} \,+\, H_{x\,x}\,H_{z\,z} \,+\, H_{y\,y}\,H_{z\,z} \,-\, H_{x\,y}^2)\},\\
H_1 =&\; \frac{1}{16}\{-3\,\delta_1^5 - 32\,\epsilon_1^2\,\delta_1^3\,\nu^2 + 8\,\delta_1^3\,(H_{x\,x} \,+\, H_{y\,y} \,+\, H_{z\,z}) + 64\epsilon_1^2\,\nu^2\,\delta_1\,H_{z\,z}\\
&\,-\,16\,\delta_1\,(H_{x\,x}\,H_{y\,y} \,+\, H_{x\,x}\,H_{z\,z} + H_{y\,y}\,H_{z\,z} \,+\, H_{x\,y}^2)\},\\
H_0 =&\; \frac{1}{64}\{\delta_1^6 + 16\,\epsilon_1^2\,\delta_1^4\,\nu^2 - 4\,\delta_1^4\,(H_{x\,x} \,+\, H_{y\,y} \,+\, H_{z\,z})\\
&\,-\,64\,H_{z\,z}\,(\epsilon_1^2\,\nu^2\,\delta_1^2 \,+\, H_{x\,x}\,H_{y\,y} \,-\, H_{x\,y}^2)\\
&\,+\,16\,\delta_1^2\,(H_{x\,x}\,H_{y\,y} \,+\, H_{x\,x}\,H_{z\,z} + H_{y\,y}\,H_{z\,z} \,-\, H_{x\,y}^2)\}.
\end{aligned}
\tag{4.8}
$$

Using Eq. (4.7) and for the various values of the parameters used corresponding to the fixed points, we obtained, all the roots which are either positive real parts of the complex roots or at-least one root is positive real number (given in the table 1). Hence these fixed points are unstable.

TABLE 1. Nature of fixed points in $x-y-$plane for $\epsilon_1 = \epsilon_2 = 1.2$, $\delta_1 = 0.2$, $\delta_2 = 0.4$, $\epsilon = 0.1$ & h = 0.1 and $z = 0$.

Fixed Point	*Characteristic Roots*	*Nature*
L_1	$0.0999999999 \pm 3.7267476661\,i$ $0.0999999999 \pm 2.3170451564\,i$ -3.5302702204 3.7302702204	*Unstable*
$L_{2,3,4}$	$0.1000000005 \pm 3.7130355866\,i$ $0.1000000045 \pm 2.3928627398\,i$ -3.5732858994 3.7732858994	*Unstable*
L_5	$-0.7989426876 \pm 1.5739500187\,i$ $0.0999999999 \pm 1.6258174939\,i$ $0.9989426876 \pm 1.5739500187\,i$	*Unstable*

5. Projections of zero-velocity surfaces

Multiplying by $\dot{x}$ and $\dot{y}$ in first and second equations of Eq. (2.6) respectively and integrating after adding both sides of these three equations of Eq. (2.6), we obtain

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \;=\; 2\,H \,-\, C \,-\, 2\int_{t_0}^{t} \left(\frac{\partial H}{\partial\, t}\right) dt, \tag{5.1}$$

where C represents the energy constant.

If the left hand side of equation (5.1) is representing the velocity (v) of the smallest body then it can also be rewritten as

$$v^2 \;=\; 2\,H \,-\, C \,-\, 2\int_{t_0}^{t} \left(\frac{\partial H}{\partial\, t}\right) dt, \tag{5.2}$$

For the possible motions, $v^2 \geq 0$ and hence

$$2\,H \,-\, C \,-\, 2\int_{t_0}^{t} \left(\frac{\partial H}{\partial\, t}\right) dt \;\geq\; 0. \tag{5.3}$$

Using the procedure of [22] and [4], the equation (5.3) can be obtained as

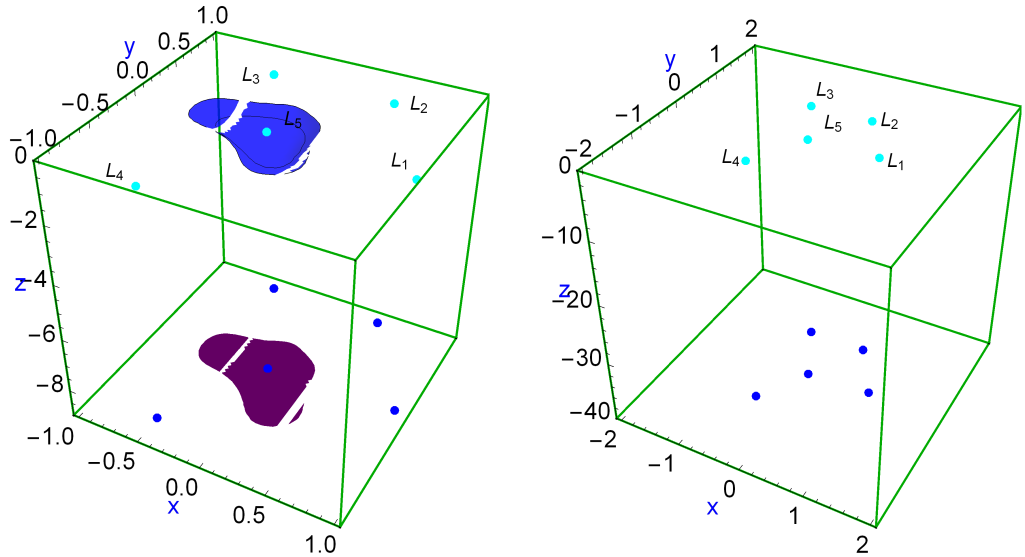
$$2\,H \;\geq\; C. \tag{5.4}$$

Utilizing equation (5.4) with software Mathematica, we can plot the zero-velocity surfaces with projections. For this firstly, we have to evaluate the value of constant C corresponding to each fixed point and then we can illustrate the surfaces. Here we have consider perturbed case ($\epsilon_2 = 1.2$, $\delta_1 = 0.2$, $\delta_2 = 0.4$, $\epsilon = 0.1$ and $h = 0.1$). In this case, we found that the value of C corresponding to fixed points $L_{1,2,3,4}$ are same while corresponding to the fixed point L_5 have different value. And finally, we have plotted the zero-velocity surfaces with projections corresponding to the fixed points ($L_{1,2,3,4}$) and L_5 and given in figures (3(a)) and (3(b)) respectively. Figure (3(a)) shows that smallest body can move everywhere except near the fixed point L_5 while figure (3(b)) shows that it can move everywhere without any restriction.

One can illustrate the zero-velocity surfaces with projections for other cases also.

6. Poincaré surfaces of section

The Poincaré surfaces of section (PSS) is an important dynamical property of the motion of the smallest body. To perform this, we have to evaluate the positions (x, y) and velocities $(\dot{x}, \dot{y})$ for different time and initial values of the motion of the smallest body in the phase space. Then we illustrate the figure between $(x, \dot{x})$ at $y = 0$, whenever the orbit intersects the plane for $\dot{y} > 0$. Here we have obtained the Poincaré surfaces of section for the various values of the parameters used and given in figures (4(a)), (4(b)), (4(c)) and (4(d)). We observed from here that there are no chaos. The Poincaré surfaces of sections in two cases of figures (4(b)) and (4(d)) are symmetrical about abscissa while the



(a) Corresponding to $L_{1,2,3,4}$ (b) Corresponding to L_5

Figure 3. Projections of zero-velocity surfaces for $\epsilon_2 = 1.2$, $\delta_1 = 0.2$, $\delta_2 = 0.4$, $\epsilon = 0.1$ and $h = 0.1$

other two cases of figures (4(a)) and (4(c)) are nearly symmetrical about abscissa
.

7. Conclusion

In this problem, we have investigated the effects of various perturbations (oblate heterogeneous shapes, spin about their own axes, coriolis force, variable mass and centrifugal forces) on the motions of the smallest body in the cyclic kite configuration. Numerical studies shows that there are five fixed points in the unperturbed case and at most nine fixed points in the perturbed cases. We also observed that the four fixed points are near by the primaries while one fixed point is near by the origin. The examination of stability shown that the fixed points here are unstable. The quasi-Jacobi integral revealed the zero-velocity surfaces with projection. In this figure shaded region shown the prohibited region while the white regions are allowed regions for motion. The Poincar e surfaces of section have no chaos and the surfaces are symmetrical and nearly symmetrical. Finally we observed that these perturbations have good enough impact on the motion of the smallest body.

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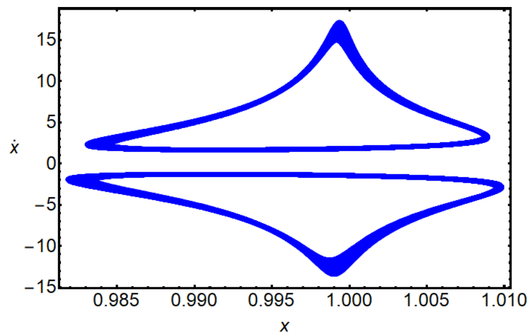
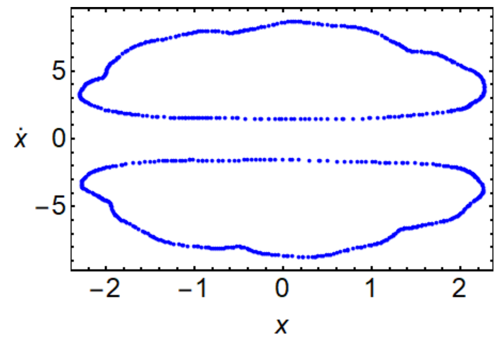
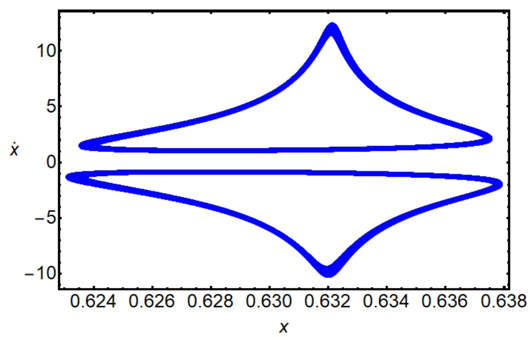


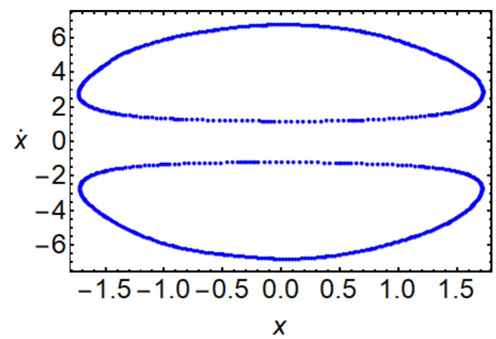
(a) For $\epsilon_1 = \epsilon_2 = 1$, $\delta_1 = 0$, $\delta_2 = 1$, $\epsilon = 0$ and h $= 0$



(b) For $\epsilon_1 = \epsilon_2 = 1.2$, $\delta_1 = 0$, $\delta_2 = 1$, $\epsilon = 0.1$ and h $= 0.1$



(c) For $\epsilon_1 = \epsilon_2 = 1.2$, $\delta_1 = 0.2$, $\delta_2 = 0.4$, $\epsilon = 0$ and h $= 0$



(d) For $\epsilon_1 = \epsilon_2 = 1.2$, $\delta_1 = 0.2$, $\delta_2 = 0.4$, $\epsilon = 0.1$ and h $= 0.1$

FIGURE 4. Poincaré surfaces of section

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