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# CHARACTERIZATION OF THE SOLUTION OF THE DIOPHANTINE EQUATION $X^2 + Y^2 = 2Z^2$

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ABSTRACT. In this paper. We are intersted to characterize the solutions of the diophantine equation $x^2 + y^2 = 2z^2$ by using the arithmetic technicals.

## 1. Introduction

The diophantine equation is intersted by a lot of Mathematicians. As Bennett characterize the solutions of the diophantine equation $x^{2n} + y^{2n} = z^5$ [2] in 2004, Frits Beukers studied the diophantine equation $Ax^p + By^q = Cz^r$ [1] in 1998 and Nils Bruin [3] search the solution of the diophantine equation $x^9 + y^8 = z^3$ and $xyz = 0$ in 1999. In our paper we are intersted to search the solution of the diophantine equation $x^2 + y^2 = 2z^2$.

## 2. Characterization of The Solution of The Diophantine Equation $x^2 + y^2 = 3z^2$

In this section we show the following result which characterizes the solution of the diophantine equation $x^2 + y^2 = 3z^2$.

**Theorem 2.1.** *The diophantine equation $x^2 + y^2 = 3z^2$ has only one solution $(0, 0, 0)$ in $\mathbb{Z}^3$.*

*Proof.* Consider the set of triples $(x_0, y_0, z_0)$ solutions of the equation $x^2+y^2 = 3z^2$ such that $x_0y_0 \neq 0$. Assume that $d = x_0 \wedge y_0$, then $x_0 = dx_1$, $y_0 = y_1d$ and $x_1 \wedge y_1 = 1$ and since $(x_0, y_0, z_0)$ is the solution of the equation $x^2 + y^2 = 3z^2$ then $z_0 = z_1d$ and the triplet $(x_1, y_1, z_1)$ is solution of the diophantine equation $x^2+y^2 = 3z^2$. So we have two cases, the first one is $x_1 \equiv 0 mod 3$ and the second is $x_1 \equiv 1 mod 3$ or $x_1 \equiv -1 mod 3$. The first case implies that $x_1^2 \equiv 0 mod 3$ and since $x_1 \wedge y_1 = 1$ then $y_1 \equiv 1 mod 3$ or $y_1 \equiv -1 mod 3$ which is implies that $y_1^2 \equiv 1 mod 3$. We deduce that $x_1^2 + y_1^2 \equiv 1 mod 3$ which is absurd because $x_1^2 + y_1^2 = 3z_1^2$. And the second case implies that $x_1^2 \equiv 1 mod 3$ We have $y_1^2 \equiv 1 mod 3$ or $y_1^2 \equiv 0 mod 3$ this implies that $x_1^2 + y_1^2 \equiv 1 mod 3$ or $x_1^2 + y_1^2 \equiv 2 mod 3$ which is absurd because $x_1^2 + y_1^2 = 3z_1^2$. ∎

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## 3. Characterization of The Solution of The Diophantine Equation $x^2+y^2=2z^2$

In this paragraph we solve the diophantine equation $x^2+y^2=2z^2$ in $\mathbb{Z}^3$.

**Theorem 3.1.** *Let the diophantine equation* $(E): x^2+y^2=2z^2$ *then the following properties are equivalents:*
**(i)** $(x,y,z)$ *is the solution of* $(E)$ *and* $x \wedge y = 1$
**(ii)** $\mid x \mid = \mid -b^2+2bc+c^2 \mid$, $\mid y \mid = \mid b^2+2bc-c^2 \mid$, $\mid z \mid = \mid b^2+c^2 \mid$ *or* $\mid x \mid = \frac{\mid -b^2+2bc+c^2 \mid}{2}$, $\mid y \mid = \frac{\mid b^2+2bc-c^2 \mid}{2}$, $\mid z \mid = \frac{\mid b^2+c^2 \mid}{2}$

*Proof.* **(ii)** $\Longrightarrow$ **(i)** We have:

$$\begin{array}{rcc} x^2+y^2 & = & (-b^2+2bc+c^2)^2+(b^2+2bc-c^2)^2 \\ & = & (2bc-(b^2-c^2))^2+(2bc+(b^2-c^2))^2 \\ & = & 2(4(bc)^2+(b^2-c^2)^2) \\ & = & 2(b^2+c^2)^2 \\ & = & 2z^2 \end{array}$$

**(i)** $\Longrightarrow$ **(ii)**
If $y$ is even and $x \wedge y = 1$ then $x$ is odd. Therefore $y^2$ is even and $x^2$ is odd then $x^2+y^2=2z^2$ is odd contradiction.
We deduce that $x, y$ are odds We have $x = 2k_1+1$ and $y = 2k_2+1$ then

$$\begin{array}{rrcc} & x^2+y^2 & = & (2k_1+1)^2+(2k_1+1)^2 \\ & & = & 4k_1^2+4k_1+1+4k_2^2+4k_2+1 \\ & & = & 4(k_1^2+k_1+k_2^2+k_2)+2 \\ & & = & 2z^2 \\ \textit{which is implies that} & z^2 & = & 2(k_1^2+k_1+k_2^2+k_2)+1 \end{array}$$

Then $z$ is odd.

Assume that $x^2 > z^2 > y^2$ then $\mid x \mid > \mid x \mid > \mid y \mid$. Let set $z \wedge x = d$. Since $y^2 = 2z^2 - x^2$ then $d^2$ divide $y^2$ so $y = kd$ which is implies that $d$ divides $x \wedge y = 1$ then $x \wedge z = 1$.
We can deduce that $z \wedge x = 1$ and $z \wedge y = 1$ $\quad (a_0)$
It is obvious that $\frac{|x|-|z|}{2}, \frac{|x|+|z|}{2}, \frac{|z|-|y|}{2}, \frac{|z|+|y|}{2} \in \mathbb{Z}$
Let set $\frac{|x|-|z|}{2} \wedge \frac{|x|+|z|}{2} = d$ then $d$ divides $\frac{|x|-|z|}{2} + \frac{|x|+|z|}{2} = \mid x \mid$ and $d$ divides $-\frac{|x|-|z|}{2} + \frac{|x|+|z|}{2} = \mid z \mid$ which is implies that $d$ divides $\mid x \mid \wedge \mid z \mid = 1$
so $\frac{|x|-|z|}{2} \wedge \frac{|x|+|z|}{2} = 1 \quad (a_1)$
Let set $\frac{|z|-|y|}{2} \wedge \frac{|z|+|y|}{2} = d$ then $d$ divides $\frac{|z|-|y|}{2} + \frac{|z|+|y|}{2} = \mid z \mid$ and $d$ divides $-\frac{|z|-|y|}{2} + \frac{|z|+|y|}{2} = \mid y \mid$ which is implies that $d$ divides $\mid z \mid \wedge \mid y \mid = 1$
so $\frac{|z|-|y|}{2} \wedge \frac{|z|+|y|}{2} = 1 \quad (a_2)$
We put

$$\begin{array}{rcll}\frac{|x|-|z|}{2}\wedge\frac{|z|-|y|}{2} & = & a>0 & (\mathbf{b_1})\\ \frac{|x|-|z|}{2}\wedge\frac{|z|+|y|}{2} & = & b>0 & (\mathbf{b_2})\\ \frac{|x|+|z|}{2}\wedge\frac{|z|-|y|}{2} & = & c>0 & (\mathbf{b_3})\\ \frac{|x|+|z|}{2}\wedge\frac{|z|+|y|}{2} & = & d>0 & (\mathbf{b_4})\end{array}$$

Since $x^2-z^2=z^2-y^2$, $(a_1)$ and $(a_2)$ show that:

$$\begin{array}{rcll}\frac{|x|-|z|}{2} & = & ab & (\mathbf{a_3})\\ \frac{|x|+|z|}{2} & = & cd & (\mathbf{a_4})\\ \frac{|z|-|y|}{2} & = & ac & (\mathbf{a_5})\\ \frac{|z|+|y|}{2} & = & bd & (\mathbf{a_6})\end{array}$$

We have $|x|>|z|>|y|>$ then $\frac{|x|+|z|}{2}>\frac{|z|+|y|}{2}$ so $cd>bd$ therefore $c>b$ it means $c-b>0 \quad (b_5)$

$$\begin{array}{rcl}|x| & = & ab+cd\\ |y| & = & bd-ac\\ |z| & = & cd-ab\\ |z| & = & ac+bd\end{array}$$

then $cd-ab=ac+bd \Longrightarrow cd-bd=ac+ab$ then

$$d(c-b)=a(b+c)$$

$1^{st}case$

$c-b$ is odd since $(a_1)$, $(a_3)$, $(a_4)$ then $a\wedge d=1$ and $b\wedge c=1$ then $c-b\wedge c+b=1$ because $c-b$ is odd since $c-b>0, a>0c>b>0$ we deduce that

$$d=b+c$$
$$a=c-b$$

then

$$\begin{array}{rcl}|x| & = & ab+cd=-b^2+2cb+c^2\\ |y| & = & bd-ac=b^2+2cb-c^2\\ |z| & = & c^2+b^2\end{array}$$

$2^{nd}case$

$c-b$ is even then $\frac{c-b}{2}\wedge\frac{c+b}{2}=1$ because $b\wedge c=1$ and $a\wedge d=1$ Since $c-b>0, a>0$ $c>b>0$ we deduce that

$$d=\frac{c+b}{2}$$
$$a=\frac{c-b}{2}$$

then

$$\begin{array}{rcl}|x| & = & ab+cd=\frac{-b^2+2cb+c^2}{2}\\ |y| & = & bd-ac=\frac{b^2+2cb-c^2}{2}\\ |z| & = & \frac{c^2+b^2}{2}\end{array}$$

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