

ON ADEQUATE RINGS: THE BASIC PROPERTIES

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ABSTRACT. In this paper, we investigate the transfer of notion of adequate rings to direct product of rings and homomorphic image. Our aim is to give new classes of commutative rings satisfying this property.

1. INTRODUCTION AND PRELIMINARIES

All rings in this paper are commutative with unity. We denote by $U(R)$ the set of unit of a ring R . And, if $a, b \in R$, $a|b$ means a divides b , that is $b = ac$ for some $c \in R$.

We know that an elementary divisor ring is a Hermite ring. Kaplansky showed that for the class of adequate domains being a Hermite ring was equivalent to being an elementary divisor ring. Gillman and Henriksen showed that this was also true for rings with zero-divisors. See for instance [3, 6, 9, 11].

Now, we give the definition of adequate ring.

Definition 1.1. A ring A is said an adequate ring if for all $a \in A - \{0\}$ and $b \in A$, there exists two non-zero elements r, s of A such that :

- a) $a = rs$.
- b) $rA + bA = A$.
- c) $\forall t \in A - U(A) : t \text{ divides } s \text{ implies } tA + bA \neq A$.

The notion of an adequate domain was originally defined by Helmer [6]. By definition, every adequate domain is a Prüfer domain. Also, every principal ideal domain is adequate. An example of an adequate ring which is not a principal ideal domain is furnished by the set of integral functions with coefficients in a field F . Also, it is clear to see that a local ring is adequate. For instance, see [6, 11].

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In this paper, we investigate the transfer of notion of adequate rings to direct product of rings and homomorphic image. Our aim is to give new classes of commutative rings satisfying this property.

2. MAIN RESULTS

Now, we investigate the transfer of the adequate property to finite direct products.

Theorem 2.1. *Let $(A_i)_{i=1,\dots,n}$ be a family of commutative rings. Then $\prod_{i=1}^{i=n} A_i$ is an adequate ring if and only if the following conditions hold:*

- 1) A_i is an adequate ring for each $i = 1, \dots, n$.
- 2) For every $i = 1, 2, \dots, n$ and every $p, q \in A_i - U(A_i)$, $pA_i + qA_i \neq A_i$.

Before proving this Theorem, we need the following Lemmas.

Lemma 2.2. *Let $(A_i)_{i=1,\dots,n}$ be a family of commutative rings. Then :*

- 1) Let $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$, and $z = (z_1, z_2, \dots, z_n)$, $\in \prod_{i=1}^{i=n} A_i$. Then, $x + y = z$ if and only if $x_i + y_i = z_i$ for every $i = 1, 2, \dots, n$. Also, $xy = z$ if and only if $x_i y_i = z_i$ for every $i = 1, 2, \dots, n$.
- 2) Let $a = (a_1, a_2, \dots, a_n) \in \prod_{i=1}^{i=n} A_i$ and $b = (b_1, b_2, \dots, b_n) \in \prod_{i=1}^{i=n} A_i$. Then, b divides a if and only if b_i divides a_i for every $i = 1, 2, \dots, n$.

Proof. Straightforward □

Lemma 2.3. *Let A be a principal ideal domain. Then A is local if and only if $pA + qA \neq A$ for every $p, q \in A - U(A)$.*

Proof. Straightforward □

Proof of Theorem 2.1

Assume that $\prod_{i=1}^{i=n} A_i$ is an adequate ring.
Let $k = 1, 2, \dots, n$. Let $a \in A_k - \{0\}$ and $b \in A_k$. For $i = 1, 2, \dots, n$, set

$$\begin{cases} u_i = v_i = 1 \in A_i & \text{if } i \neq k \\ u_k = a & \text{and } i = k \end{cases}$$

Set $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$. Since $u_k = a \neq 0$, then $a \neq 0$. Using the fact $\prod_{i=1}^{i=n} A_i$ is an adequate ring, $u \in \prod_{i=1}^{i=n} A_i - \{0\}$ and $v \in \prod_{i=1}^{i=n} A_i$. So, there exist $r = (r_1, r_2, \dots, r_n)$ and $s = (s_1, s_2, \dots, s_n) \in \prod_{i=1}^{i=n} A_i$ such that :

$$\begin{cases} u = rs \\ r(\prod_{i=1}^{i=n} A_i) + v(\prod_{i=1}^{i=n} A_i) = \prod_{i=1}^{i=n} A_i \\ \forall r(\prod_{i=1}^{i=n} A_i) - U(\prod_{i=1}^{i=n} A_i) : t \text{ divides } s \text{ implies } t(\prod_{i=1}^{i=n} A_i) + v(\prod_{i=1}^{i=n} A_i) \neq (\prod_{i=1}^{i=n} A_i) \end{cases}$$

In fact of view $u = rs$, it follows that $u_k = r_k s_k$ and so $a = r_k s_k$.
 $r(\prod_{i=1}^{i=n} A_i) + v(\prod_{i=1}^{i=n} A_i) = \prod_{i=1}^{i=n} A_i$. So, $(\prod_{i=1}^{i=n} r_i A_i) + (\prod_{i=1}^{i=n} v_i A_i) = \prod_{i=1}^{i=n} A_i$.
 Therefore, $(\prod_{i=1}^{i=n} (r_i A_i + v_i A_i)) = \prod_{i=1}^{i=n} A_i$. We obtain $r_k A_k + v_k A_k = A_k$. Hence,
 $r_k A_k + b A_k = A_k$.

Let $t \in A_k - U(A_k)$ such that t divides s_k . $\forall i = 1, 2, \dots, n$

$$\begin{cases} w_i = 1 & \text{if } i \neq k \\ w_k = t & \text{if } i = k \end{cases}$$

Let $w = (w_1, w_2, \dots, w_n)$. Since $w_k = t \in A_k - U(A_k)$, then by Lemma 2.2,
 $w = (w_1, w_2, \dots, w_n) \in \prod_{i=1}^{i=n} A_i - U(\prod_{i=1}^{i=n} A_i)$.
 $\forall i = 1, 2, \dots, n$:

$$\begin{cases} \text{If } i \neq k & w_i = 1 \text{ implies } w_i \text{ divides } s_i \\ \text{If } i = k & w_i = w_k = t \text{ implies } w_i \text{ divides } s_i \text{ since } w_k \text{ divides } s_k \end{cases}$$

By 2) of Lemma 2.2, it follows that w divides s .

Using the fact $\prod_{i=1}^{i=n} A_i$ is an adequate ring, $w \in \prod_{i=1}^{i=n} A_i - U(\prod_{i=1}^{i=n} A_i)$ and w divides s . So, $w(\prod_{i=1}^{i=n} A_i) + v(\prod_{i=1}^{i=n} A_i) \neq \prod_{i=1}^{i=n} A_i$. By 2) of Lemma 2.2,
 $w_i(\prod_{i=1}^{i=n} A_i) + v_i(\prod_{i=1}^{i=n} A_i) \neq \prod_{i=1}^{i=n} A_i$. Therefore, there exists $j = 1, 2, \dots, n$
 such that $w_j A_j + v_j A_j \neq A_j$. Suppose that $j \neq k$. Then $w_j A_j + v_j A_j = w_j A_j + 1 A_j = w_j A_j + A_j = A_j$, a contradiction. And so, $j = k$ implies
 $t A_k + b A_k = w_k A_k + v_k A_k = w_j A_j + v_j A_j \neq A_j$. Thus, A_k is an adequate ring.

Suppose that there exists a natural integer $k = 1, 2, \dots, n$ and $p, q \in A_k - U(A_k)$ such that $p A_k + q A_k = A_k$. For every $i = 1, 2, \dots, n$, set

$$\begin{cases} a_i = b_i = 1 & \text{if } i \neq k \\ a_k = 0 \in A_k & b_k = q \text{ if } i = k \end{cases}$$

Let $a = (a_1, a_2, \dots, a_n) (\neq 0)$ and $b = (b_1, b_2, \dots, b_n)$ with $n \geq 2$. Using the fact $\prod_{i=1}^{i=n} A_i$ is an adequate ring, $a \in \prod_{i=1}^{i=n} A_i - \{0\}$ and $b \in \prod_{i=1}^{i=n} A_i$, there exists
 $r = (r_1, r_2, \dots, r_n) \in \prod_{i=1}^{i=n} A_i$ such that:

$$\begin{cases} a = rs \\ r(\prod_{i=1}^{i=n} A_i) + b(\prod_{i=1}^{i=n} A_i) = \prod_{i=1}^{i=n} A_i \\ \forall t \in (\prod_{i=1}^{i=n} A_i) - U(\prod_{i=1}^{i=n} A_i) : t \text{ divides } s \text{ implies } t(\prod_{i=1}^{i=n} A_i) + b(\prod_{i=1}^{i=n} A_i) \neq \prod_{i=1}^{i=n} A_i \end{cases}$$

For every $i = 1, 2, \dots, n$, set

$$\begin{cases} t_i = 1 \in A_i & \text{if } i \neq k \\ t_k = p & \text{if } i = k \end{cases}$$

Let $t = (t_1, t_2, \dots, t_n)$, $t_k = p \notin U(A_k)$. Since $\prod_{i=1}^{i=n} A_i$ is an adequate ring, $t \in (\prod_{i=1}^{i=n} A_i) - U(\prod_{i=1}^{i=n} A_i)$ and t divides s , then $t(\prod_{i=1}^{i=n} A_i) + b(\prod_{i=1}^{i=n} A_i) \neq \prod_{i=1}^{i=n} A_i$.
 For every $i = 1, 2, \dots, n$, we have:

$$\begin{cases} t_i A_i + b_i A_i = 1 A_i + 1 A_i = A_i + A_i = A_i & \text{if } i \neq k \\ t_k A_k + b_k A_k = p A_k + q A_k = A_k & \text{if } i = k \end{cases}$$

$$\begin{aligned}
t\left(\prod_{i=1}^{i=n} A_i\right) + b\left(\prod_{i=1}^{i=n} A_i\right) &= (t_i \prod_{i=1}^{i=n} A_i) + (b_i \prod_{i=1}^{i=n} A_i) \text{ by 2) of Lemma 2.2) } \\
&= \prod_{i=1}^{i=n} t_i A_i + b_i A_i \\
&= \prod_{i=1}^{i=n} A_i.
\end{aligned}$$

which is a contradiction. Hence, $pA_i + qA_i \neq A_i$ for every $i = 1, 2, \dots, n$ and $p, q \in A_i - U(A_i)$, as desired.

Conversely, assume that 1) and 2) of Theorem 2.1 hold. Let $a = (a_1, a_2, \dots, a_n) \in \prod_{i=1}^{i=n} A_i - \{0\}$ and $b = (b_1, b_2, \dots, b_n) \in \prod_{i=1}^{i=n} A_i$. Let $i = 1, \dots, n$, such that $a_i \neq 0$. There exists $r'_i, s'_i \in A_i$ such that:

$$\left\{ \begin{array}{l} a_i = r'_i s'_i \\ r'_i A_i + b_i A_i = A_i \\ \forall t \in A_i - U(A_i) : t \text{ divides } s'_i \text{ implies } tA_i + b_i A_i \neq A_i \end{array} \right.$$

For every $i = 1, 2, \dots, n$, we have:

$$\left\{ \begin{array}{ll} r_i = r'_i \text{ and } s_i = s'_i & \text{if } a_i \neq 0 \\ r_i = 1 \text{ and } s_i = a_i = 0 & \text{if } a_i = 0 \text{ and } b_i \notin U(A_i) \\ r_i = a_i = 0 \text{ and } s_i = 1 & \text{if } a_i = 0 \text{ and } b_i \in U(A_i) \end{array} \right.$$

Set $r = (r_1, r_2, \dots, r_n)$ and $s = (s_1, s_2, \dots, s_n)$. Then:

$$\left\{ \begin{array}{ll} a_i = r'_i s'_i = r_i s_i & \text{if } a_i \neq 0 \\ a_i = 0 = 1.0 = r_i s_i & \text{if } a_i = 0 \text{ and } b_i \notin U(A_i) \\ a_i = 0.1 = r_i s_i \text{ and } s_i = 1 & \text{if } a_i = 0 \text{ and } b_i \in U(A_i) \end{array} \right.$$

So $a = rs$ and for every $i = 1, 2, \dots, n$, we have:

$$\left\{ \begin{array}{ll} r_i A_i + b_i A_i = r'_i A_i + b_i A_i = A_i & \text{if } a_i \neq 0 \\ r_i A_i + b_i A_i = 1A_i + b_i A_i = A_i & \text{if } a_i = 0 \text{ and } b_i \notin U(A_i) \\ r_i A_i + b_i A_i = 0A_i + b_i A_i = b_i A_i = A_i & \text{if } a_i = 0 \text{ and } b_i \in U(A_i) \end{array} \right.$$

Therefore,

$$\begin{aligned}
r\left(\prod_{i=1}^{i=n} A_i\right) + b\left(\prod_{i=1}^{i=n} A_i\right) &= \prod_{i=1}^{i=n} (r_i A_i) + \prod_{i=1}^{i=n} (b_i A_i) \text{ (by 2) of Lemma 2.2) } \\
&= \prod_{i=1}^{i=n} (r_i A_i + b_i A_i) \text{ (by 1) of Lemma 2.2) } \\
&= \prod_{i=1}^{i=n} A_i.
\end{aligned}$$

Set $t = (t_1, t_2, \dots, t_n) (\in (\prod_{i=1}^{i=n} A_i) - U(\prod_{i=1}^{i=n} A_i))$. Since, $t \notin U(\prod_{i=1}^{i=n} A_i)$, then there exists $k = 1, 2, \dots, n$ such that $t_k \notin U(A_k)$ and so t_k divides s_k (since t divides s). Consequently, $s_k \neq 1$ since $b_k \notin U(A_k)$. Hence, two cases are possible :

Case 1: $a_k \neq 0$.

Since t_k divides s_k , then t_k divides s'_k (since $s_k = s'_k$). So, $t_k A_k + b_k A_k \neq A_k$.

Case 2: $a_k = 0$ (and so $b_k \notin U(A_k)$).

Since $t_k, b_k \in A_k - U(A_k)$, then $t_k A_k + b_k A_k \neq A_k$. Thus, for all cases $t_k A_k + b_k A_k \neq A_k$. It follows that

$$\begin{aligned} t(\prod_{i=1}^{i=n} A_i) + b(\prod_{i=1}^{i=n} A_i) &= \prod_{i=1}^{i=n} (t_i A_i) + \prod_{i=1}^{i=n} (b_i A_i) \\ &= \prod_{i=1}^{i=n} (t_i A_i + b_i A_i) \\ &= \prod_{i=1}^{i=n} A_i. \quad (\text{since } t_k A_k + b_k A_k \neq A_k) \end{aligned}$$

Hence, $\prod_{i=1}^{i=n} A_i$ is an adequate ring.

The following corollaries are consequence of Theorem 2.1.

Corollary 2.4. *Let A be a ring and let n be a positive integer such that $n \geq 2$. Then A^n is an adequate ring if and only if the following conditions hold:*

- 1) A is an adequate ring.
- 2) $\forall p, q \in A - U(A) : pA + qA \neq A$.

Corollary 2.5. *Let n be a positive integer such that $n \geq 2$, and let $A_1, A_2, \dots, A_n$ be a principal ideal domains. Then $\prod_{i=1}^{i=n} A_i$ is an adequate ring if and only if A_i is a local ring for every $i = 1, 2, \dots, n$.*

Proof. Assume that $\prod_{i=1}^{i=n} A_i$ is an adequate ring. Then by Theorem 2.1, for every $k = 1, 2, \dots, n$, and $p, q \in A_k - U(A_k)$, $pA_k + qA_k \neq A_k$. Since A_k is a principal ideal domain, then $p, q \in A_k - U(A_k)$ and so A_k is local by Lemma 2.3.

Conversely, assume that A_i is a local ring for all $i = 1, 2, \dots, n$. Then by Corollary 2.6, $\prod_{i=1}^{i=n} A_i$ is an adequate ring. $\square$

Example 2.6. Let $(A_i)_{i=1, \dots, n}$ be a family of local commutative rings, where $i \geq 2$. Then $\prod_{i=1}^{i=n} A_i$ is a non-local adequate ring.

The following example shows that the hypothesis "local" is necessary to obtain the direct product of a principal ideal domain is an adequate ring.

Example 2.7. $\mathbb{Z}^n$ is not an adequate ring for every non negative integer $n \geq 2$.

Now, we investigate the transfer of notion of adequate ring to homomorphic image which is our second main result. To prove it, we need the following Lemmas.

Lemma 2.8. *Let A be a ring, B be a ring, $f : A \rightarrow B$ be a surjective ring homomorphism. Then :*

- (1) *Let $a, b, c \in A$. Then, $f(a) = f(b)f(c)$ if and only if $a - bc \in \ker f$.*
- (2) *Let $a, b \in A$. Then, $f(b)|f(a)$ if and only if $aA \subset bA + \ker f$.*
- (3) *Let $a \in A$. Then, $f(a) \in U(B)$ if and only if $aA + \ker f = A$.*
- (4) *Let I be an ideal of A . Then $f(I) = B$ if and only if $I + \ker f = A$.*
- (5) *Let $a, b \in A$. Then, $f(a)B + f(b)B = B$ if and only if $aA + bA + \ker f = A$.*

Proof. (1) Let $a, b, c \in A$. Then:

$$\begin{aligned}
 f(a) = f(b)f(c) &\Leftrightarrow f(a) = f(bc) \\
 &\Leftrightarrow f(a) - f(bc) = 0 \\
 &\Leftrightarrow f(a - bc) = 0 \\
 &\Leftrightarrow a - bc \in \ker f
 \end{aligned}$$

(2) Let $a, b \in A$. Then:

$$\begin{aligned}
 f(b)|f(a) &\Leftrightarrow \exists x \in B / f(a) = f(b)x \\
 &\Leftrightarrow \exists c \in A : f(a) = f(b)f(c) \text{ since } B=f(A) \\
 &\Leftrightarrow \exists c \in A : a - bc \in \ker f \\
 &\Leftrightarrow \exists c \in A \text{ and } \exists u \in \ker f : a - bc = u \\
 &\Leftrightarrow \exists c \in A \text{ and } \exists u \in \ker f : a = bc + u \\
 &\Leftrightarrow a \in bA + \ker f \\
 &\Leftrightarrow Aa \subset bA + \ker f
 \end{aligned}$$

(3) Let $a \in A$. Then, $f(a) \in U(B) \Leftrightarrow f(a)|f(1) \Leftrightarrow A = 1A \subset aA + \ker f$ (by (2) above) $\Leftrightarrow A = aA + \ker f$.

(4) Let I be an ideal of A . Then:

$$\begin{aligned}
f(I) = B &\Leftrightarrow 1 \in f(I) \Leftrightarrow \exists a \in I : f(a) = f(1) \\
&\Leftrightarrow \exists a \in I : f(a) - f(1) = f(a - 1) = 0 \\
&\Leftrightarrow \exists a \in I : a - 1 \in \ker f \Leftrightarrow \exists u \in \ker f : u = a - 1 \\
&\Leftrightarrow \exists a \in I : a - 1 \in \ker f \Leftrightarrow \exists u \in \ker f : a - u = 1 \\
&\Leftrightarrow \exists a \in I : \text{ and } \exists v \in \ker f : a + v = 1 \\
&\Leftrightarrow 1 \in I + \ker f \\
&\Leftrightarrow I + \ker f = A
\end{aligned}$$

(5) Let $a, b \in A$. Then, $f(a)B + f(b)B = B \Leftrightarrow f(a)f(A) + f(b)(A) = B \Leftrightarrow f(aA + bA) = B \Leftrightarrow aA + bA + \ker f$ (by (4) above).

□

Lemma 2.9. *Let A be a ring, B be a ring, $f : A \rightarrow B$ be a surjective ring homomorphism. Then B is an adequate ring if and only if the following conditions hold :*

- (1) $a - rs \in \ker f$.
- (2) $rA + bA + \ker f = A$.
- (3) $\forall t \in A : sA \subset tA + \ker f \neq A \Rightarrow tA + bA + \ker f \neq A$.

Proof. Assume that B is an adequate ring. Let $a \in A - \ker f$ and let $b \in A$. since $a \notin \ker f$, then $f(a) \neq 0$. Using the fact B is an adequate ring, $f(a) \in B - \{0\}$ and let $f(b) \in B$, then there exists $u, v \in B$ such that

$$\begin{cases} f(a) = uv \\ uB + f(b)B = B \\ \forall k \in B - U(B) : k|v \Rightarrow kB + f(b)B \neq B. \end{cases}$$

Since $u, v \in B$, then there exists $r, s \in A$ such that $u = f(r)$ and $v = f(s)$ since $B = f(A)$. Then:

- (1) $f(a) = uv = f(r)f(s) \Rightarrow a - rs \in \ker f$ (by (1) of Proposition 2.8).
- (2) $f(a)B + f(b)B = uB + vB = B$. So, by (5) of Lemma 2.8, $aA + bA + \ker f = A$.
- (3) Let $t \in A$ such that $sA \subset tA + \ker f \neq A$. Since $tA + \ker f \neq A$, then by (3) of Lemma 2.8, $f(t) \in B - U(B)$. By assumption, $sA \subset tA + \ker f \neq A$, and so by (2) of Proposition 2.8, $f(t)|f(s)$. Therefore, $f(t)|v$. Since $f(t) \in B - U(B)$ and $f(t)|v$, then $f(t)B + f(b)B \neq B$. Consequently, by (5) of Lemma 2.8, $tA + bA + \ker f \neq A$.

Conversely, assume that for every $a \in A - \ker f$ and $b \in A$, there exists $r, s \in A$ such that (1), (2) and (3) hold. Let $x \in B - \{0\}$ and let $y \in B$. Then $\exists a, b \in A$ such that $x = f(a)$ and $y = f(b)$ (since $B = f(A)$). We have $f(a) = x \neq 0 \Rightarrow a \notin \ker f \Rightarrow a \in A - \ker f$. Since $a \in A - \ker f$ and $b \in A$, then there exists $r, s \in A$ such that:

- $(a - rs) \in \ker f$.
- $rA + bA + \ker f = A$.
- $\forall t \in A : sA \subset tA + \ker f \neq A \Rightarrow tA + bA + \ker f \neq A$.

We have $a - rs \in \ker f$ and so by (1) of Proposition 2.8, $x = f(a) = f(r)f(s)$. In fact of view, $rA + bA + \ker f = A$, by (5) of Lemma 2.8, $f(r)B + f(b)B = B$. Hence, $f(r)B + yB = B$.

- Let $k \in B - U(B)$ such that $k|s$. Since $k \in B = f(A)$, then there exists $t \in A$ such that $k = f(t)$. Using the fact $f(t) = k \notin U(B)$, then by (3) of Proposition 2.8, $tA + \ker f \neq A$. We have $v|k \Rightarrow f(t)|f(s)$ and so $sA \subset tA + \ker f \neq A$. Hence, $tA + bA + \ker f \neq A$. By (5) of Lemma 2.8, it follows that $kB + yB = f(t)B + f(b)B \neq B$. Thus, B is an adequate ring. $\square$

Now, we are able to give our second main result.

Theorem 2.10. *Let A be a principal ideal domain, B be a ring and $f : A \rightarrow B$ be a surjective ring homomorphism. Then B is an adequate ring.*

Proof. Assume that A is a principal ideal domain. Then, A is an adequate ring. Let $a \in A - \ker f$ and let $b \in A$. Hence, there exists $d \in A$ such that $bA + \ker f = dA$ (since A is a principal ideal domain), where $a \in A - \{0\}$ (since $a \in A - \ker f$) and $d \in A$ and so there exists $r, s \in A$ such that:

$$\begin{cases} a = rs \\ rA + dA = A \\ \forall t \in A - U(A) : t|s \Rightarrow tA + dA \neq A. \end{cases}$$

Since $a = rs$, then $a - rs = 0 \in \ker f$. So, r and s verify the statements (1) of Lemma 2.9 for each $a, b \in A$. We also have $rA + dA = A$ and so $rA + bA + \ker f = A$ (since $da = ba + \ker f$). Therefore, r and s verify statement (2) of Lemma 2.9.

Let $t \in A$ such that $sA \subset tA + \ker f \neq A$. Since A is a principal ideal domain, then there exists $k \in A$ such that $tA + \ker f = kA$ and so:

$$\begin{cases} kA = tA + \ker f \neq A \Rightarrow k \notin U(A) \Rightarrow k \in A - U(A) \\ sA \subset tA + \ker f \Rightarrow sA \subset kA \Rightarrow k|s. \end{cases}$$

Since $k \in A - U(A)$ and k divides s , then $kA + dA \neq A$. It follows that $tA + bA + \ker f = tA + \ker f + bA + \ker f = kA + dA \neq A$ and so r and s verify the statement (3) of Lemma 2.9. Hence, B is an adequate ring, by Proposition 2.9 and this completes the proof of Theorem . $\square$

As a consequence of Theorem 2, we obtain the following Corollary.

Corollary 2.11. *Let A be a principal ideal domain. Then, A/aA is an adequate ring for every $a \in A - U(A)$.*

Example 2.12. $\mathbb{Z}/n\mathbb{Z}$ is an adequate ring for every $n \in \mathbb{Z} - U(\mathbb{Z})$.

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