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SOME RESULTS ON THE DIFFERENCE BETWEEN CONSECUTIVE PERFECT POWERS

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This paper is dedicated to Professor Dr. Alfredo Novelli

Abstract. Let P_n be the n-th perfect power. In this article we prove various theorems on the difference $d_n = P_{n+1} - P_n$. In the literature there does not exist an article dedicated to a general study on the difference d_n between consecutive perfect powers. This article is the first attempt to study it.

1. Introduction and Preliminary Results

A natural number of the form m^n where m is a positive integer and $n \geq 2$ is called a perfect power. The first few terms of the integer sequence of perfect powers are

$$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128 \ldots$$

If P_n denotes the n-th perfect power and d_n denotes the difference $P_{n+1} - P_n$ then

$$\begin{aligned} d_1 &= P_2 - P_1 = 4 - 1 = 3, \\ d_2 &= P_3 - P_2 = 8 - 4 = 4, \\ d_3 &= P_4 - P_3 = 9 - 8 = 1, \\ d_4 &= P_5 - P_4 = 16 - 9 = 7, \\ &\vdots \end{aligned}$$

In this article we prove various theorems on the difference $d_n = P_{n+1} - P_n$.

There exist some important conjectures and results on d_n. The Pillai's conjecture establish the following limit

$$\lim_{n\to\infty} d_n = \infty.$$

This is an unsolved problem.

On the other hand, P. Mihăilescu (see [3], [4] and [5]) proved that the only pair of consecutive perfect powers is 8 and 9, thus proving Catalan's conjecture.

We have (see [2]) the following lemma.

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Lemma 1.1. *Let p_h $(h \geq 3)$ be the h-th prime. We have the following asymptotic formula*

$$P_n = n^2 + \sum_{i=1}^{m} c_i n^{g_i} + f(n) n^{1+\frac{2}{p_h}}, \tag{1.1}$$

where $f(n) \to -2$, $2 > g_1 = 5/3 > g_2 > \ldots > g_m > 1 + \frac{2}{p_h}$ and the c_i are rational coefficients.

For example, if $h = 3$ then (see [2])

$$P_n = n^2 - 2n^{5/3} + f(n)n^{7/5}.$$

If $h = 4$ then (see [2])

$$P_n = n^2 - 2n^{5/3} - 2n^{7/5} + \frac{13}{3}n^{4/3} + f(n)n^{9/7}.$$

If $h = 5$ then (see [2])

$$P_n = n^2 - 2n^{5/3} - 2n^{7/5} + \frac{13}{3}n^{4/3} - 2n^{9/7} + 2n^{6/5} - 2n^{13/11} + o\left(n^{13/11}\right).$$

We also need the following lemmas. In the first lemma we find an upper bound for d_n, namely $2n$. We shall see that this upper bound is the least upper bound of the form An, where A is a positive real number.

Lemma 1.2. *We have the following inequality*

$$d_n = P_{n+1} - P_n < 2n \qquad (n \geq 3).$$

Proof. We have $P_1 = 1^2 = 1$, $P_2 = 2^2 = 4$, $P_3 = 8$ and $P_4 = 3^2$. Consequently if $n \geq 4$ then $P_n \leq (n-1)^2$, since the squares are perfect powers. Therefore if $n \geq 4$ we have

$$P_4 \leq (A-1)^2 \leq P_n < P_{n+1} \leq A^2 \leq n^2.$$

Therefore

$$P_{n+1} - P_n \leq A^2 - (A-1)^2 \leq n^2 - (n-1)^2 = 2n - 1 < 2n.$$

Besides $P_4 - P_3 = 9 - 8 = 1 < 2.3$. □

It is an open problem to obtain a lower bound for the differences d_n from a certain value of n. However, we shall prove that fn $(0 < f < 2)$ is a lower bound for almost all the differences d_n. Perhaps $d_n \sim 2n$. This is a more strong conjecture that the Pillai's conjecture.

Lemma 1.3. *Let $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ be two series of positive terms such that $\lim_{i\to\infty} \frac{a_i}{b_i} = 1$. If $\sum_{i=1}^{\infty} b_i$ is divergent then we have the following limit*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} = 1.$$

Proof. See for example [6, page 332]. □

Lemma 1.4. *Let us consider the function*

$$h(x,y)=\frac{A+y}{B+x} \qquad (0<A<B)$$

defined in the closed triangular region $D=\{(x,y):0\le y\le x, 0\le x\le a\}$ $(a>0)$. *We have the following inequality* $((x,y)\in D)$

$$\left|\frac{A+y}{B+x}-\frac{A}{B}\right|\le\frac{a}{B+a}.$$

Proof. Since $h(x,y)$ is continuous in the bounded closed triangular region D, it has (Weierstrass's Theorem) an absolute maximum and an absolute minimum on D. We have

$$\frac{\partial h}{\partial x}=\frac{-A-y}{(B+x)^2},$$
$$\frac{\partial h}{\partial y}=\frac{1}{B+x}>0.$$

Consequently the absolute maximum and the absolute minimum are in the triangular frontier.

Let us consider the segment $(x,y)=(x,x)$ $x\in[0,a]$. We have

$$\begin{gathered} h(x,x)=\frac{A+x}{B+x}, \\ h(0,0)=\frac{A}{B}, \qquad h(a,a)=\frac{A+a}{B+a}, \\ h'(x,x)=\frac{B-A}{(B+x)^2}>0. \end{gathered} \tag{1.2}$$

Let us consider the segment $(x,y)=(x,0)$ $x\in[0,a]$. We have

$$\begin{gathered} h(x,0)=\frac{A}{B+x}, \\ h(0,0)=\frac{A}{B}, \qquad h(a,0)=\frac{A}{B+a}, \\ h'(x,0)=\frac{-A}{(B+x)^2}<0. \end{gathered} \tag{1.3}$$

Let us consider the segment $(x,y)=(a,y)$ $y\in[0,a]$. We have

$$\begin{gathered} h(a,y)=\frac{A+y}{B+a}, \\ h(a,0)=\frac{A}{B+a}, \qquad h(a,a)=\frac{A+a}{B+a}, \\ h'(a,y)=\frac{1}{B+a}>0. \end{gathered} \tag{1.4}$$

Consequently (see (1.2), (1.3) and (1.4)) we have

$$\frac{A}{B+a}<\frac{A}{B}<\frac{A+a}{B+a}.$$

Therefore the absolute maximum is $\frac{A+a}{B+a}$ and the absolute minimum is $\frac{A}{B+a}$.

Hence we have the inequality ($(x, y) \in D$)

$$\frac{A}{B+a} \leq \frac{A+y}{B+x} \leq \frac{A+a}{B+a},$$

and (see above) the inequality

$$\frac{A}{B+a} < \frac{A}{B} < \frac{A+a}{B+a}.$$

Hence

$$\left|\frac{A+y}{B+x} - \frac{A}{B}\right| \leq \frac{A+a}{B+a} - \frac{A}{B+a} = \frac{a}{B+a}.$$

□

2. The Difference $d_n = P_{n+1}$-P_n in Short Intervals

$\lfloor x \rfloor$ denotes the largest integer that does not exceed x.

Theorem 2.1. *If $0 < \theta \leq 1/3$ and $0 < \epsilon < \theta$ are two fixed real numbers then we have the following asymptotic formula*

$$P_{n+\lfloor n^\theta \rfloor} - P_n = 2n \lfloor n^\theta \rfloor + o\left(n^{1+\epsilon}\right). \tag{2.1}$$

Proof. We have (see (1.1))

$$P_n = n^2 + \sum_{i=1}^{m} c_i n^{g_i} + f(n) n^{1+\frac{2}{p_h}}. \tag{2.2}$$

Consequently

$$\begin{aligned} P_{n+\lfloor n^\theta \rfloor} &= \left(n + \lfloor n^\theta \rfloor\right)^2 + \sum_{i=1}^{m} c_i \left(n + \lfloor n^\theta \rfloor\right)^{g_i} \\ &+ f\left(n + \lfloor n^\theta \rfloor\right)\left(n + \lfloor n^\theta \rfloor\right)^{1+\frac{2}{p_h}}. \end{aligned} \tag{2.3}$$

On the other hand, we have

$$\left(n + \lfloor n^\theta \rfloor\right)^2 - n^2 = 2n \lfloor n^\theta \rfloor + \lfloor n^\theta \rfloor^2, \tag{2.4}$$

and (mean value Theorem)

$$\begin{aligned} \left(n + \lfloor n^\theta \rfloor\right)^{g_i} - n^{g_i} &= g_i \left(n + \epsilon_i(n) \lfloor n^\theta \rfloor\right)^{g_i - 1} \lfloor n^\theta \rfloor \\ &= g_i n^{g_i - 1 + \theta} \left(1 + \frac{\epsilon_i(n) \lfloor n^\theta \rfloor}{n}\right)^{g_i - 1} \frac{\lfloor n^\theta \rfloor}{n^\theta} \\ &= g_i(n) n^{g_i - 1 + \theta}, \end{aligned} \tag{2.5}$$

where $0 < \epsilon_i(n) < 1$ and $g_i(n) \to g_i$. Besides we have

$$\left(n + \lfloor n^\theta \rfloor\right)^{1+\frac{2}{p_h}} = n^{1+\frac{2}{p_n}} \left(1 + \frac{\lfloor n^\theta \rfloor}{n}\right)^{1+\frac{2}{p_h}} = h(n) n^{1+\frac{2}{p_h}}, \tag{2.6}$$

where $h(n) \to 1$. Equations (2.2), (2.3), (2.4), (2.5) and (2.6) give

$$\begin{aligned} P_{n+\lfloor n^\theta \rfloor} - P_n &= 2n \lfloor n^\theta \rfloor + \lfloor n^\theta \rfloor^2 + \sum_{i=1}^{m} c_i g_i(n) n^{g_i - 1 + \theta} \\ &+ f\left(n + \lfloor n^\theta \rfloor\right) h(n) n^{1+\frac{2}{p_h}} - f(n) n^{1+\frac{2}{p_h}} \\ &= 2n \lfloor n^\theta \rfloor + \lfloor n^\theta \rfloor^2 + \sum_{i=1}^{m} c_i g_i(n) n^{g_i - 1 + \theta} \\ &+ o\left(n^{1+\frac{2}{p_h}}\right) = 2n \lfloor n^\theta \rfloor + \frac{\lfloor n^\theta \rfloor^2}{n^{1+\frac{2}{p_h}}} \, n^{1+\frac{2}{p_h}} \\ &+ \sum_{i=1}^{m} c_i g_i(n) n^{g_i - 2 + \theta - \frac{2}{p_h}} \, n^{1+\frac{2}{p_h}} + o\left(n^{1+\frac{2}{p_h}}\right) \\ &= 2n \lfloor n^\theta \rfloor + o\left(n^{1+\frac{2}{p_h}}\right) = 2n \lfloor n^\theta \rfloor + o\left(n^{1+\epsilon}\right), \end{aligned}$$

since $0 < \theta \leq 1/3$ and the greater g_i is $g_1 = 5/3$ (see Lemma 1.1). Besides we can choose p_h such that $\frac{2}{p_h} < \epsilon$. □

Corollary 2.2. *Note that (see (2.1))*

$$P_{n+\lfloor n^\theta \rfloor} - P_n = (P_{n+1} - P_n) + (P_{n+2} - P_{n+1}) + \ldots + \left(P_{n+\lfloor n^\theta \rfloor} - P_{n+\lfloor n^\theta \rfloor - 1}\right),$$

where there are $\lfloor n^\theta \rfloor$ consecutive differences. The mean difference is asymptoticaly equal to $2n$. That is

$$\frac{P_{n+\lfloor n^\theta \rfloor} - P_n}{\lfloor n^\theta \rfloor} \sim 2n.$$

Proof. We have (Theorem 2.1)

$$P_{n+\lfloor n^\theta \rfloor} - P_n = 2n \lfloor n^\theta \rfloor + o\left(n^{1+\epsilon}\right) = 2n \lfloor n^\theta \rfloor + o(n) n^\epsilon.$$

Consequently

$$\frac{P_{n+\lfloor n^\theta \rfloor} - P_n}{\lfloor n^\theta \rfloor} = 2n + o(n) \frac{n^\epsilon}{n^\theta} \frac{n^\theta}{\lfloor n^\theta \rfloor} = 2n + o(n) o(1) = 2n + o(n)$$

Since

$$\frac{n^\theta}{\lfloor n^\theta \rfloor} \to 1, \quad \frac{n^\epsilon}{n^\theta} \to 0$$

□

Theorem 2.3. *Let us consider the $\lfloor n^\theta \rfloor$ consecutive differences*

$$d_n = (P_{n+1} - P_n), d_{n+1} = (P_{n+2} - P_{n+1}), \ldots, d_{n+\lfloor n^\theta \rfloor - 1} = \left(P_{n+\lfloor n^\theta \rfloor} - P_{n+\lfloor n^\theta \rfloor - 1}\right)$$

where $n \geq 3$. We have

$$d_i < 2\left(n + \lfloor n^\theta \rfloor - 1\right) = 2n + 2\lfloor n^\theta \rfloor - 2 \qquad \left(i = n, n+1, \ldots, n + \lfloor n^\theta \rfloor - 1\right).$$

Proof. It is an immediate consequence of Lemma 1.2. □

Theorem 2.4. *Let $0 < f < 2$ a fixed real number. Let us consider the $\lfloor n^\theta \rfloor$ consecutive differences*

$$d_n = (P_{n+1} - P_n), d_{n+1} = (P_{n+2} - P_{n+1}), \ldots, d_{n+\lfloor n^\theta \rfloor - 1} = \left(P_{n+\lfloor n^\theta \rfloor} - P_{n+\lfloor n^\theta \rfloor - 1}\right).$$

Let $k(n)$ be the number of these differences such that $d_i \leq fn$. We have the following limit

$$\lim_{n\to\infty} \frac{k(n)}{\lfloor n^\theta \rfloor} = 0.$$

Proof. We have

$$0 \leq \frac{k(n)}{\lfloor n^\theta \rfloor} \leq 1.$$

Hence

$$0 \leq \liminf \frac{k(n)}{\lfloor n^\theta \rfloor} \leq \limsup \frac{k(n)}{\lfloor n^\theta \rfloor} \leq 1.$$

We shall prove that

$$\limsup \frac{k(n)}{\lfloor n^\theta \rfloor} = 0.$$

Therefore

$$\liminf \frac{k(n)}{\lfloor n^\theta \rfloor} = \limsup \frac{k(n)}{\lfloor n^\theta \rfloor} = 0.$$

Consequently, we have the desired limit

$$\lim_{n\to\infty} \frac{k(n)}{\lfloor n^\theta \rfloor} = 0.$$

First, suppose that

$$\limsup \frac{k(n)}{\lfloor n^\theta \rfloor} = h,$$

where $0 < h < 1$.

Therefore there exists a strictly increasing subsequence n_j $(n_1 < n_2 < n_3 < \cdots)$ such that

$$\lim_{j\to\infty} \frac{k(n_j)}{\lfloor n_j^\theta \rfloor} = h.$$

Therefore

$$k(n_j) = h \lfloor n_j^\theta \rfloor + o_1\left(\lfloor n_j^\theta \rfloor\right),$$

where

$$o_1\left(\lfloor n_j^\theta \rfloor\right) = a(j) \lfloor n_j^\theta \rfloor \qquad (a(j) \to 0),$$

and hence

$$\sum_{d_i \leq fn_j} d_i \leq k(n_j) f n_j = hf \lfloor n_j^\theta \rfloor n_j + f o_1\left(\lfloor n_j^\theta \rfloor\right) n_j. \tag{2.7}$$

Theorem 2.1 gives

$$\sum_{d_i \leq fn_j} d_i + \sum_{d_i > fn_j} d_i = P_{n_j + \lfloor n_j^\theta \rfloor} - P_{n_j} = 2n_j \lfloor n_j^\theta \rfloor + o\left(n_j^{1+\epsilon}\right). \tag{2.8}$$

Let $q(n_j)$ be the number of differences such that $d_i > fn_j$. We have

$$\begin{aligned} q(n_j) &= \lfloor n_j^\theta \rfloor - k(n_j) = \lfloor n_j^\theta \rfloor - h \lfloor n_j^\theta \rfloor - o_1 \left(\lfloor n_j^\theta \rfloor \right) \\ &= (1-h) \lfloor n_j^\theta \rfloor - o_1 \left(\lfloor n_j^\theta \rfloor \right). \end{aligned} \tag{2.9}$$

Equations (2.7) and (2.8) give

$$\begin{aligned} \sum_{d_i > fn_j} d_i &\geq 2n_j \lfloor n_j^\theta \rfloor + o\left(n_j^{1+\epsilon}\right) - hf \lfloor n_j^\theta \rfloor n_j - fo_1 \left(\lfloor n_j^\theta \rfloor \right) n_j \\ &= \frac{2-hf}{1-h} n_j \left((1-h) \lfloor n_j^\theta \rfloor - o_1 \left(\lfloor n_j^\theta \rfloor \right) \right) \\ &+ \frac{2-hf}{1-h} n_j o_1 \left(\lfloor n_j^\theta \rfloor \right) + o\left(n_j^{1+\epsilon}\right) - fo_1 \left(\lfloor n_j^\theta \rfloor \right) n_j \\ &= \frac{2-hf}{1-h} n_j \left((1-h) \lfloor n_j^\theta \rfloor - o_1 \left(\lfloor n_j^\theta \rfloor \right) \right) + o\left(n_j \lfloor n_j^\theta \rfloor \right). \end{aligned}$$

That is

$$\sum_{d_i > fn_j} d_i \geq \frac{2-hf}{1-h} n_j \left((1-h) \lfloor n_j^\theta \rfloor - o_1 \left(\lfloor n_j^\theta \rfloor \right) \right) + o\left(n_j \lfloor n_j^\theta \rfloor \right), \tag{2.10}$$

where $\frac{2-hf}{1-h} > 2$. Let us consider the following equation (see (2.10))

$$\begin{aligned} &\frac{2-hf}{1-h} n_j \left((1-h) \lfloor n_j^\theta \rfloor - o_1 \left(\lfloor n_j^\theta \rfloor \right) \right) + o\left(n_j \lfloor n_j^\theta \rfloor \right) \\ = &\frac{2-hf}{1-h} n_j \left((1-h) \lfloor n_j^\theta \rfloor - o_1 \left(\lfloor n_j^\theta \rfloor \right) \right) A(n_j). \end{aligned} \tag{2.11}$$

Equation (2.11) gives

$$A(n_j) = 1 + \frac{o\left(n_j \lfloor n_j^\theta \rfloor \right)}{\frac{2-hf}{1-h} n_j \left((1-h) \lfloor n_j^\theta \rfloor - o_1 \left(\lfloor n_j^\theta \rfloor \right) \right)} = 1 + o(1). \tag{2.12}$$

Equations (2.10), (2.11), (2.12) and (2.9) give

$$\begin{aligned} \sum_{d_i > fn_j} d_i &\geq \frac{2-hf}{1-h} A(n_j) n_j \left((1-h) \lfloor n_j^\theta \rfloor - o_1 \left(\lfloor n_j^\theta \rfloor \right) \right) = \frac{2-hf}{1-h} A(n_j) n_j q(n_j) \\ &\geq B n_j q(n_j) \qquad (B > 2). \end{aligned} \tag{2.13}$$

Consequently in the $q(n_j)$ differences there are differences that satisfy the inequality (see (2.13))

$$d_i \geq B n_j \qquad (B > 2). \tag{2.14}$$

On the other hand, Theorem 2.3 establish

$$d_i < 2n_j + 2 \lfloor n_j^\theta \rfloor - 2 \qquad \left(i = n_j, n_j + 1, \ldots, n_j + \lfloor n_j^\theta \rfloor - 1 \right). \tag{2.15}$$

Clearly (2.14) and (2.15) are an evident contradiction.

Second, suppose that

$$\limsup \frac{k(n)}{\lfloor n^\theta \rfloor} = 1.$$

Therefore there exists a strictly increasing subsequence n_j ($n_1 < n_2 < n_3 < \cdots$) such that

$$\lim_{j\to\infty} \frac{k(n_j)}{\lfloor n_j^{\theta} \rfloor} = 1.$$

Hence

$$k(n_j) = \lfloor n_j^{\theta} \rfloor - o\left(\lfloor n_j^{\theta} \rfloor\right),$$

and consequently

$$\sum_{d_i \leq f n_j} d_i \leq k(n_j) f n_j = f \lfloor n_j^{\theta} \rfloor n_j - f o\left(\lfloor n_j^{\theta} \rfloor\right) n_j. \tag{2.16}$$

On the other hand, we have

$$q(n_j) = \lfloor n_j^{\theta} \rfloor - k(n_j) = o\left(\lfloor n_j^{\theta} \rfloor\right) \leq \alpha \lfloor n_j^{\theta} \rfloor \qquad \left(0 < \alpha < \frac{2-f}{2}\right). \tag{2.17}$$

Equations (2.8), (2.16) and (2.17) give

$$\begin{aligned}
\sum_{d_i > f n_j} d_i &\geq 2n_j \lfloor n_j^{\theta} \rfloor + o\left(n_j^{1+\epsilon}\right) - f \lfloor n_j^{\theta} \rfloor n_j + f o\left(\lfloor n_j^{\theta} \rfloor\right) n_j \\
&= (2-f) n_j \lfloor n_j^{\theta} \rfloor + o\left(n_j^{1+\epsilon}\right) + f o\left(\lfloor n_j^{\theta} \rfloor\right) n_j \\
&= \left(\frac{2-f}{\alpha} + o(1)\right) n_j \alpha \lfloor n_j^{\theta} \rfloor \geq C n_j \alpha \lfloor n_j^{\theta} \rfloor \\
&\geq C n_j q(n_j) \qquad (C > 2).
\end{aligned} \tag{2.18}$$

Therefore in the $q(n_j)$ differences there are differences that satisfy the inequality (see (2.18))

$$d_i \geq C n_j \qquad (C > 2). \tag{2.19}$$

Clearly (2.15) and (2.19) are an evident contradiction.

Hence

$$\limsup \frac{k(n)}{\lfloor n^{\theta} \rfloor} = 0.$$

□

Corollary 2.5. *Let $0 < f < 2$ a fixed real number. Let us consider the $\lfloor n^{\theta} \rfloor$ consecutive differences*

$$d_n = (P_{n+1} - P_n), d_{n+1} = (P_{n+2} - P_{n+1}), \ldots, d_{n+\lfloor n^{\theta} \rfloor - 1} = \left(P_{n+\lfloor n^{\theta} \rfloor} - P_{n+\lfloor n^{\theta} \rfloor - 1}\right).$$

Let $p(n)$ be the number of these differences such that $d_i > fn$. We have the following limit

$$\lim_{n\to\infty} \frac{p(n)}{\lfloor n^{\theta} \rfloor} = 1.$$

Proof. It is an immediate consequence of Theorem 2.4. Since $k(n) + p(n) = \lfloor n^{\theta} \rfloor$. □

3. The Sequence of Differences P_{n+1}-P_n

Theorem 3.1. *Let $0 < f < 2$ a fixed real number. Let us consider the first n consecutive differences*

$$d_1 = (P_2 - P_1), d_2 = (P_3 - P_2), \ldots, d_n = (P_{n+1} - P_n).$$

Let $v(n)$ be the number of these differences such that $d_i > fi$. We have the following limit

$$\lim_{n\to\infty} \frac{v(n)}{n} = 1. \tag{3.1}$$

Proof. If $d_i > fi$ $(i = 1, 2, 3, \ldots)$ then the theorem is proved.

Suppose that there exist differences such that $d_i \le fi$. Suppose that in the set $d_1, d_2, \ldots, d_{n_0-1}$ there are differences such that $d_i \le fi$, where n_0^θ is large.

Now, let us consider the following set partition of the set of differences $d_i = P_{i+1} - P_i$ $(i \ge n_0)$.

The first set (0-set) contain the following $\lfloor n_0^\theta \rfloor$ differences.

$$d_{n_0} = P_{n_0+1} - P_{n_0}, d_{n_0+1} = P_{n_0+2} - P_{n_0+1}, \ldots, d_{n_0+\lfloor n_0^\theta \rfloor -1} = P_{n_0+\lfloor n_0^\theta \rfloor} - P_{n_0+\lfloor n_0^\theta \rfloor -1}$$

If we put $n_1 = n_0 + \lfloor n_0^\theta \rfloor$ the second set (1-set) contain the following $\lfloor n_1^\theta \rfloor$ differences.

$$d_{n_1} = P_{n_1+1} - P_{n_1}, d_{n_1+1} = P_{n_1+2} - P_{n_1+1}, \ldots, d_{n_1+\lfloor n_1^\theta \rfloor -1} = P_{n_1+\lfloor n_1^\theta \rfloor} - P_{n_1+\lfloor n_1^\theta \rfloor -1}$$

If we put $n_2 = n_1 + \lfloor n_1^\theta \rfloor$ the 2-set contain the following $\lfloor n_2^\theta \rfloor$ differences.

$$d_{n_2} = P_{n_2+1} - P_{n_2}, d_{n_2+1} = P_{n_2+2} - P_{n_2+1}, \ldots, d_{n_2+\lfloor n_2^\theta \rfloor -1} = P_{n_2+\lfloor n_2^\theta \rfloor} - P_{n_2+\lfloor n_2^\theta \rfloor -1}$$

In general, if we put $n_k = n_{k-1} + \lfloor n_{k-1}^\theta \rfloor$ the k-set contain the following $\lfloor n_k^\theta \rfloor$ differences.

$$d_{n_k} = P_{n_k+1} - P_{n_k}, d_{n_k+1} = P_{n_k+2} - P_{n_k+1}, \ldots, d_{n_k+\lfloor n_k^\theta \rfloor -1} = P_{n_k+\lfloor n_k^\theta \rfloor} - P_{n_k+\lfloor n_k^\theta \rfloor -1}$$

Note that the sequence of positive integers n_i $(i = 0, 1, 2, \ldots)$ is strictly increasing, therefore $\lim_{i\to\infty} n_i = \infty$. Consequently the sequence n_i^θ $(i = 0, 1, 2, \ldots)$ is also strictly increasing and $\lim_{i\to\infty} n_i^\theta = \infty$. Besides $\lfloor n_{i+1}^\theta \rfloor \ge \lfloor n_i^\theta \rfloor$.

Let us consider the k-set. Let $s(k)$ be the number of differences d_i $(i = n_k, n_k + 1, \ldots, n_k + \lfloor n_k^\theta \rfloor - 1)$ such that $d_i \le fi$. Let $r(k)$ be the number of differences d_i $(i = n_k, n_k+1, \ldots, n_k + \lfloor n_k^\theta \rfloor - 1)$ such that $d_i \le f(n_k + \lfloor n_k^\theta \rfloor - 1)$. Let $t(k)$ be the number of differences d_i $(i = n_k, n_k + 1, \ldots, n_k + \lfloor n_k^\theta \rfloor - 1)$ such that $d_i \le f' n_k$ $(f < f' < 2)$. Clearly $s(k) \le r(k)$. We have

$$\lim_{k\to\infty} \frac{f(n_k + \lfloor n_k^\theta \rfloor - 1)}{f n_k} = \lim_{k\to\infty} \frac{n_k + \lfloor n_k^\theta \rfloor - 1}{n_k} = 1.$$

Consequently

$$f(n_k + \lfloor n_k^\theta \rfloor - 1) < f' n_k,$$

and hence $s(k) \le r(k) \le t(k)$. Theorem 2.4 gives

$$\lim_{k\to\infty} \frac{t(k)}{\lfloor n_k^\theta \rfloor} = 0.$$

Therefore
$$\lim_{k\to\infty}\frac{s(k)}{\lfloor n_k^{\theta}\rfloor}=0.$$
Let us consider the k-set. Let $u(k)$ be the number of differences d_i $(i=n_k,n_k+1,\ldots,n_k+\lfloor n_k^{\theta}\rfloor-1)$ such that $d_i>fi$. Since $s(k)+u(k)=\lfloor n_k^{\theta}\rfloor$ the following limit holds
$$\lim_{k\to\infty}\frac{u(k)}{\lfloor n_k^{\theta}\rfloor}=1. \tag{3.2}$$
Let C be the number of differences d_i $(1\leq i\leq n_0-1)$ such that $d_i>fi$. We have (see above) $C<n_0-1$. Equation (3.2) and Lemma 1.3 give (see the enunciate of Theorem 3.1)
$$\begin{aligned}&\lim_{s\to\infty}\frac{C+u(0)+u(1)+\cdots+u(s)}{n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor}\\ =\ &\lim_{s\to\infty}\frac{v(n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor)}{n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor}=1. \end{aligned}\tag{3.3}$$
That is, we have obtained a subsequence where limit (3.1) is fulfilled.

Let $x_{s+1}=1,2,\ldots,\lfloor n_{s+1}^{\theta}\rfloor-1$ and let y_{s+1} be the number of differences d_i in the $(s+1)$-set such that the following two conditions hold, $d_i>fi$ and $i=n_{s+1},n_{s+1}+1,\ldots,n_{s+1}+x_{s+1}-1$. Therefore
$$v\left(n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor+x_{s+1}\right)=C+u(0)+u(1)+\cdots+u(s)+y_{s+1}$$
Lemma 1.4 gives (see the former formula and equation (3.3))
$$\begin{aligned}&\left|\frac{v\left(n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor+x_{s+1}\right)}{n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor+x_{s+1}}\right.\\ -\ &\left.\frac{v\left(n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor\right)}{n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor}\right|\\ =\ &\left|\frac{C+u(0)+u(1)+\cdots+u(s)+y_{s+1}}{n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor+x_{s+1}}\right.\\ -\ &\left.\frac{C+u(0)+u(1)+\cdots+u(s)}{n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor}\right|\\ \leq\ &\frac{\lfloor n_{s+1}^{\theta}\rfloor}{n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor+\lfloor n_{s+1}^{\theta}\rfloor}. \end{aligned}\tag{3.4}$$
We have
$$\begin{aligned}&\frac{\lfloor n_{s+1}^{\theta}\rfloor}{n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor+\lfloor n_{s+1}^{\theta}\rfloor}\\ =\ &\frac{1}{1+\frac{n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor}{\lfloor n_{s+1}^{\theta}\rfloor}}. \end{aligned}\tag{3.5}$$

On the other hand we have the inequalities

$$\frac{n_0 - 1 + \lfloor n_0^\theta \rfloor + \lfloor n_1^\theta \rfloor + \cdots + \lfloor n_s^\theta \rfloor}{\lfloor n_{s+1}^\theta \rfloor} \geq \frac{\lfloor n_{s-k}^\theta \rfloor + \lfloor n_{s-k+1}^\theta \rfloor + \cdots + \lfloor n_{s-1}^\theta \rfloor + \lfloor n_s^\theta \rfloor}{\lfloor n_{s+1}^\theta \rfloor}$$

$$\geq \frac{-(k+1) + n_{s-k}^\theta + n_{s-k+1}^\theta + \cdots + n_{s-2}^\theta + n_{s-1}^\theta + n_s^\theta}{n_{s+1}^\theta}. \tag{3.6}$$

Note that

$$n_{i+1}^\theta = \left(n_i + \lfloor n_i^\theta \rfloor\right)^\theta = n_i^\theta \left(1 + \frac{\lfloor n_i^\theta \rfloor}{n_i}\right)^\theta.$$

Consequently

$$\frac{n_s^\theta}{n_{s+1}^\theta} = \frac{1}{\left(1 + \frac{\lfloor n_s^\theta \rfloor}{n_s}\right)^\theta} \to 1 \qquad (s \to \infty),$$

$$\frac{n_{s-1}^\theta}{n_{s+1}^\theta} = \frac{n_{s-1}^\theta}{n_s^\theta \left(1 + \frac{\lfloor n_s^\theta \rfloor}{n_s}\right)^\theta} = \frac{1}{\left(1 + \frac{\lfloor n_{s-1}^\theta \rfloor}{n_{s-1}}\right)^\theta \left(1 + \frac{\lfloor n_s^\theta \rfloor}{n_s}\right)^\theta} \to 1 \qquad (s \to \infty),$$

$$\vdots$$

$$\frac{n_{s-k}^\theta}{n_{s+1}^\theta} = \frac{1}{\left(1 + \frac{\lfloor n_{s-k}^\theta \rfloor}{n_{s-k}}\right)^\theta \cdots \left(1 + \frac{\lfloor n_{s-1}^\theta \rfloor}{n_{s-1}}\right)^\theta \left(1 + \frac{\lfloor n_s^\theta \rfloor}{n_s}\right)^\theta} \to 1 \qquad (s \to \infty).$$

Therefore (see (3.6))

$$\frac{n_0 - 1 + \lfloor n_0^\theta \rfloor + \lfloor n_1^\theta \rfloor + \cdots + \lfloor n_s^\theta \rfloor}{\lfloor n_{s+1}^\theta \rfloor} \geq k.$$

Since k is arbitrarily large we have

$$\lim_{s \to \infty} \frac{n_0 - 1 + \lfloor n_0^\theta \rfloor + \lfloor n_1^\theta \rfloor + \cdots + \lfloor n_s^\theta \rfloor}{\lfloor n_{s+1}^\theta \rfloor} = \infty. \tag{3.7}$$

Equations (3.5) and (3.7) give

$$\lim_{s \to \infty} \frac{\lfloor n_{s+1}^\theta \rfloor}{n_0 - 1 + \lfloor n_0^\theta \rfloor + \lfloor n_1^\theta \rfloor + \cdots + \lfloor n_s^\theta \rfloor + \lfloor n_{s+1}^\theta \rfloor} = 0. \tag{3.8}$$

Let $\epsilon > 0$ be. There exists s_0 such that if $s \geq s_0$ we have (see (3.3))

$$1 - \frac{\epsilon}{2} < \frac{v(n_0 - 1 + \lfloor n_0^\theta \rfloor + \lfloor n_1^\theta \rfloor + \cdots + \lfloor n_s^\theta \rfloor)}{n_0 - 1 + \lfloor n_0^\theta \rfloor + \lfloor n_1^\theta \rfloor + \cdots + \lfloor n_s^\theta \rfloor} < 1 + \frac{\epsilon}{2}. \tag{3.9}$$

and (see (3.4) and (3.8))

$$\begin{aligned}&\left|\frac{v\left(n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor+x_{s+1}\right)}{n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor+x_{s+1}}\right.\\ &-\left.\frac{v\left(n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor\right)}{n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor}\right|<\frac{\epsilon}{2}\end{aligned} \tag{3.10}$$

Equation (3.10) gives

$$\begin{aligned}&\frac{v\left(n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor\right)}{n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor}-\frac{\epsilon}{2}\\ <\ &\frac{v\left(n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor+x_{s+1}\right)}{n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor+x_{s+1}}\\ <\ &\frac{v\left(n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor\right)}{n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor}+\frac{\epsilon}{2}\end{aligned} \tag{3.11}$$

Equations (3.11) and (3.9) give

$$1-\epsilon<\frac{v\left(n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor+x_{s+1}\right)}{n_0-1+\lfloor n_0^{\theta}\rfloor+\lfloor n_1^{\theta}\rfloor+\cdots+\lfloor n_s^{\theta}\rfloor+x_{s+1}}<1+\epsilon \tag{3.12}$$

Consequently there exists a such that if $n\geq a$ we have (see (3.9) and (3.12))

$$1-\epsilon<\frac{v(n)}{n}<1+\epsilon.$$

That is (see (3.1))

$$\lim_{n\to\infty}\frac{v(n)}{n}=1.$$

□

Corollary 3.2. *Let $0<f<2$ a fixed real number. Let us consider the first n consecutive differences*

$$d_1=(P_2-P_1),d_2=(P_3-P_2),\ldots,d_n=(P_{n+1}-P_n).$$

Let $w(n)$ be the number of these differences such that $d_i\leq fi=(2-\epsilon)i$. We have the following limit

$$\lim_{n\to\infty}\frac{w(n)}{n}=0.$$

Proof. It is an immediate consequence of Theorem 3.1, since $v(n)+w(n)=n$. □

The following corollary is other version of Theorem 3.1. Note that $\epsilon>0$ can be arbitrarily small.

Corollary 3.3. *Let us consider the first n consecutive differences*

$$d_1=(P_2-P_1),d_2=(P_3-P_2),\ldots,d_n=(P_{n+1}-P_n).$$

Let $v(n)$ be the number of these differences such that $(2-\epsilon)i < d_i < 2i$ $(\epsilon > 0)$. We have the following limit

$$\lim_{n\to\infty} \frac{v(n)}{n} = 1.$$

Proof. Theorem 3.1 and Lemma 1.2. □

Theorem 3.4. *We have*

$$\sum_{i=1}^{\infty} \frac{1}{d_i} = \infty.$$

Proof. It is an immediate consequence of Lemma 1.2 and the comparison test. □

Theorem 3.5. *We have*

$$\sum_{i=1}^{n} \frac{1}{d_i} = o(n).$$

Proof. Let $0 < f < 2$ a fixed real number. Let us consider the first n consecutive differences

$$d_1 = (P_2 - P_1), d_2 = (P_3 - P_2), \ldots, d_n = (P_{n+1} - P_n).$$

Let $v(n)$ be the number of these differences such that $d_i > fi$. We have the following limit (Theorem 3.1)

$$\lim_{n\to\infty} \frac{v(n)}{n} = 1.$$

Let $w(n)$ be the number of these differences such that $1 \leq d_i \leq fi$. We have the following limit (Corollary 3.2)

$$\lim_{n\to\infty} \frac{w(n)}{n} = 0.$$

Consequently

$$\sum_{i=1}^{n} \frac{1}{d_i} = \sum_{d_i > fi} \frac{1}{d_i} + \sum_{1\leq d_i \leq fi} \frac{1}{d_i} \leq \sum_{i=1}^{n} \frac{1}{fi} + \sum_{1\leq d_i\leq fi} \frac{1}{1} = g(n)\log n + w(n) = o(n),$$

where $g(n) \to 1/f$. Note that we have use the well-known asymptotic formula $\sum_{i=1}^{n} \frac{1}{i} \sim \log n$. □

Theorem 3.5 has the following corollary.

Corollary 3.6. *Let d be a difference between two consecutive perfect powers, that is, there exist i such that $d_i = d$. Let n_d be the number of times that d repeat in the first n consecutive differences $d_1, d_2, \ldots, d_n$. The following limit holds*

$$\lim_{n\to\infty} \frac{n_d}{n} = 0.$$

That is, any difference d between two consecutive perfect powers has zero density.

Proof. Suppose that $\frac{n_d}{n}$ has not limit zero. Consequently there exist $\epsilon > 0$ such that $\frac{n_d}{n} \geq \epsilon$ for infitite values of n. Consequently we have for these infinite values of n

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{d_i} = \frac{1}{n}\left(\sum_{\substack{d_i=d\\ d_i\leq d_n}}\frac{1}{d} + \sum_{\substack{d_i\neq d\\ d_i\leq d_n}}\frac{1}{d_i}\right) = \frac{n_d}{n}\frac{1}{d} + \frac{1}{n}\sum_{\substack{d_i\neq d\\ d_i\leq d_n}}\frac{1}{d_i} \geq \frac{\epsilon}{d}$$

This is an evident contradiction (see Theorem 3.5). Consequently

$$\lim_{n\to\infty}\frac{n_d}{n} = 0.$$

□

Question: Is there some difference d repeated infinite times?

Theorem 3.7. *Let k be a fixed positive integer. We have the following asymptotic formula*

$$\sum_{i=1}^{n} d_i^k \sim \frac{2^k}{k+1}n^{k+1} \tag{3.13}$$

Proof. If $k = 1$ the theorem is true since (see equation (1.1)) $P_n \sim n^2$. Suppose $k \geq 2$. We have

$$\sum_{i=1}^{n} i^k = \int_0^n x^k \, dx + O(n^k) = \frac{n^{k+1}}{k+1} + O(n^k) \sim \frac{n^{k+1}}{k+1}$$

Note that the left side is a sum of areas of rectangles of basis 1 and height i^k. Therefore, the integral aproximates this sum with error $O(n^k)$ since x^k is a function strictly increasing.

Therefore we have

$$\sum_{i=1}^{n} i^k = g(n)\frac{n^{k+1}}{k+1} \tag{3.14}$$

where $g(n) \to 1$.

Let $\epsilon > 0$ arbitrarily small.

Lemma 1.2 and equation (3.14) give

$$\sum_{i=1}^{n} d_i^k \leq \sum_{i=1}^{n}(2i)^k = 2^k\sum_{i=1}^{n} i^k = g(n)\frac{2^k}{k+1}n^{k+1} < (1+\epsilon)\frac{2^k}{k+1}n^{k+1} \tag{3.15}$$

Let f be such that $\left(\frac{f}{2}\right)^k > 1-\epsilon$. Theorem 3.1, Corollary 3.2 and equation (3.14) give

$$\begin{aligned}
\sum_{i=1}^{n} d_i^k &= \sum_{\substack{i\le n\\ d_i>fi}} d_i^k + \sum_{\substack{i\le n\\ d_i\le fi}} d_i^k \ge \sum_{\substack{i\le n\\ d_i>fi}} (fi)^k + \sum_{\substack{i\le n\\ d_i\le fi}} 1^k = f^k \sum_{\substack{i\le n\\ d_i>fi}} i^k + w(n)\\
&\ge f^k \sum_{i=1}^{v(n)} i^k + w(n) = f^k g(v(n)) \frac{v(n)^{k+1}}{k+1} + w(n)\\
&= \frac{f^k}{k+1} g(v(n)) \left(\frac{v(n)}{n}\right)^{k+1} n^{k+1} + w(n) = \frac{f^k}{k+1} h(n) n^{k+1}\\
&= \left(\frac{f}{2}\right)^k h(n) \frac{2^k}{k+1} n^{k+1} > (1-\epsilon)\frac{2^k}{k+1} n^{k+1} \qquad (3.16)
\end{aligned}$$

where $h(n) \to 1$.

Equations (3.15) and (3.16) give

$$1-\epsilon < \frac{\sum_{i=1}^{n} d_i^k}{\frac{2^k}{k+1} n^{k+1}} < 1+\epsilon$$

□

In the following theorem and its corollary we prove certain simetry in the distribution of differences.

Theorem 3.8. *Let m be a fixed positive integer such that $m \ge 2$. Let r be a fixed nonnegative integer such that $r \in \{0, 1, \ldots, m-1\}$. We have the following asymptotic formula*

$$\sum_{i=1}^{n} d_{mi-r} \sim mn^2 \sim \frac{1}{m} P_{mn} \qquad (3.17)$$

Proof. Let $0 < f < 2$ a fixed real number. Let us consider the n differences

$$d_{mi-r} \qquad (i = 1, 2, \ldots, n)$$

Let $w_{m,r}(n)$ be the number of these differences such that $d_{mi-r} \le f(mi-r)$. Let $v_{m,r}(n)$ be the number of these differences such that $d_{mi-r} > f(mi-r)$. Note that the union the m sets of these differences ($r = 0, 1, \ldots, m-1$) is the set of the first mn consecutive differences.

We have (Corollary 3.2)

$$\lim_{n\to\infty} \frac{w(mn)}{mn} = 0.$$

Therefore

$$\lim_{n\to\infty} \frac{w(mn)}{n} = 0.$$

Now

$$w_{m,r}(n) \le w(mn)$$

Consequently we have the following limit

$$\lim_{n\to\infty}\frac{w_{m,r}(n)}{n}=0. \tag{3.18}$$

Limit (3.18) implies the following limit

$$\lim_{n\to\infty}\frac{v_{m,r}(n)}{n}=1. \tag{3.19}$$

Since $w_{m,r}(n)+v_{m,r}(n)=n$.

Let $\epsilon>0$ arbitrarily small.

Lemma 1.2 and equation (3.14) give

$$\sum_{i=1}^{n} d_{mi-r}\leq\sum_{i=1}^{n}2(mi-r)\leq 2m\sum_{i=1}^{n} i=g(n)mn^2<(1+\epsilon)mn^2 \tag{3.20}$$

Let f be such that $\frac{f}{2}>1-\epsilon$. Equations (3.18) , (3.19) and (3.14) give

$$\begin{aligned}
\sum_{i=1}^{n} d_{mi-r} &= \sum_{\substack{i\leq n\\ d_{mi-r}>f(mi-r)}} d_{mi-r}+\sum_{\substack{i\leq n\\ d_{mi-r}\leq f(mi-r)}} d_{mi-r}\\
&\geq \sum_{\substack{i\leq n\\ d_{mi-r}>f(mi-r)}} d_{mi-r}+\sum_{\substack{i\leq n\\ d_{mi-r}\leq f(mi-r)}} 1\\
&= \sum_{\substack{i\leq n\\ d_{mi-r}>f(mi-r)}} d_{mi-r}+w_{m,r}(n)\geq\sum_{i=1}^{v_{m,r}(n)} f(mi-r)+w_{m,r}(n)\\
&= fm\sum_{i=1}^{v_{m,r}(n)} i-frv_{m,r}(n)+w_{m,r}(n)=fmg(v_{m,r}(n))\frac{(v_{m,r}(n))^2}{2}\\
&- frv_{m,r}(n)+w_{m,r}(n)=\frac{f}{2}mn^2g(v_{m,r}(n))\frac{(v_{m,r}(n))^2}{n^2}\\
&- mn^2fr\frac{v_{m,r}(n)}{mn^2}+mn^2\frac{w_{m,r}(n)}{mn^2}=\frac{f}{2}h(n)mn^2>(1-\epsilon)mn^2
\end{aligned} \tag{3.21}$$

where $h(n)\to 1$. Equations (3.20) and (3.21) give

$$1-\epsilon<\frac{\sum_{i=1}^{n} d_{mi-r}}{mn^2}<1+\epsilon$$

That is, equation (3.17). Note that $P_{mn}\sim(mn)^2$. Since $P_n\sim n^2$ (see equation (1.1)). □

Corollary 3.9. *We have the following limit*

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n} d_{mi-r_1}}{\sum_{i=1}^{n} d_{mi-r_2}}=1 \qquad (r_1\neq r_2)$$

Proof. It is an immediate consequence of Theorem 3.8. □

The following two theorems are well-known (see [1]).

Theorem 3.10. *Let us consider the n open intervals $(0, 1^2), (1^2, 2^2), \ldots, ((n-1)^2, n^2)$. Let $S(n)$ be the number of these n open intervals that contain some perfect power. The following limit holds,*

$$\lim_{n\to\infty} \frac{S(n)}{n} = 0.$$

Therefore, almost all the open intervals are empty.

Theorem 3.11. *Let us consider the n open intervals $(0, 1^2), (1^2, 2^2), \ldots, ((n-1)^2, n^2)$. Let $F(n)$ be the number of perfect powers in these n open intervals. The following asymptotic formula holds*

$$F(n) \sim n^{\frac{2}{3}}.$$

In this article, we shall prove a more precise theorem. Before, we need the following lemma.

Lemma 3.12. *Let us consider the sequence of disjoint intervals*

$$\big[k^2, (k+1)^2\big) \qquad (k = 0, 1, 2, \ldots) \tag{3.22}$$

If n is a positive integer such that $n \geq 2$ then n^3 and $(n+1)^3$ pertain to different and not consecutive intervals of the sequence (3.22).

Proof. We have

$$\big\lfloor n^{3/2} \big\rfloor \leq n^{3/2} < \big\lfloor n^{3/2} \big\rfloor + 1$$

Therefore

$$\big\lfloor n^{3/2} \big\rfloor^2 \leq n^3 < \big(\big\lfloor n^{3/2} \big\rfloor + 1\big)^2$$

In the same way we obtain

$$\big\lfloor (n+1)^{3/2} \big\rfloor^2 \leq (n+1)^3 < \big(\big\lfloor (n+1)^{3/2} \big\rfloor + 1\big)^2$$

If $n \geq 2$ then we have the inequality

$$\frac{4}{3} < \sqrt{n}$$

Now

$$\begin{aligned}
\frac{4}{3} < \sqrt{n} \quad &\Rightarrow \quad 4n^{3/2} < 3n^2 \Rightarrow n^3 + 4n^{3/2} + 4 < n^3 + 3n^2 + 3n + 1 \\
&\Rightarrow \quad \big(n^{3/2} + 2\big)^2 < \big((n+1)^{3/2}\big)^2 \Rightarrow n^{3/2} + 2 < (n+1)^{3/2} \\
&\Rightarrow \quad n^{3/2} + 1 < (n+1)^{3/2} - 1
\end{aligned}$$

That is, if $n \geq 2$ then we have the inequality

$$n^{3/2} + 1 < (n+1)^{3/2} - 1 \tag{3.23}$$

Now, we have the inequalities

$$\big\lfloor n^{3/2} \big\rfloor + 1 \leq n^{3/2} + 1 \tag{3.24}$$

$$(n+1)^{3/2} - 1 < \big\lfloor (n+1)^{3/2} \big\rfloor \tag{3.25}$$

Equations (3.23), (3.24) and (3.25) give

$$\big\lfloor n^{3/2} \big\rfloor + 1 < \big\lfloor (n+1)^{3/2} \big\rfloor$$

□

We now prove the following more precise theorem (see Theorem 3.10)

Theorem 3.13. *Let us consider the n open intervals $(0, 1^2), (1^2, 2^2), \ldots, ((n-1)^2, n^2)$. Let $S(n)$ be the number of these n open intervals that contain some perfect power. We have the following asymptotic formula*

$$S(n) \sim n^{2/3}.$$

Let $S_1(n)$ be the number of these n open intervals that contain only one perfect power. We have the following asymptotic formula

$$S_1(n) \sim n^{2/3}.$$

Let $S_2(n)$ be the number of these n open intervals that contain at least two perfect powers. We have the following formula

$$S_2(n) = o(n^{2/3}).$$

Proof. Let us consider the inequality $i^3 \leq n^2$. This inequality has the solutions

$$1, 2, \ldots, \lfloor n^{2/3} \rfloor$$

Let us consider the inequality $i^6 \leq n^2$. This inequality has the solutions

$$1, 2, \ldots, \lfloor n^{1/3} \rfloor$$

Consequently the number of cubes in the n open intervals $(0, 1^2), (1^2, 2^2), \ldots, ((n-1)^2, n^2)$ is

$$\left(\lfloor n^{2/3} \rfloor - \lfloor n^{1/3} \rfloor\right) \sim n^{2/3}$$

Therefore (Lemma 3.12 and Theorem 3.11) we have

$$S_1(n) \sim n^{2/3}.$$

and

$$S_2(n) = o(n^{2/3}).$$

Finally we have

$$S(n) \sim n^{2/3}.$$

Since $S(n) = S_1(n) + S_2(n)$. □

Corollary 3.14. *We have the following asymptotic formula*

$$S(n) \sim S_1(n)$$

Proof. It is an immediate consequence of Theorem 3.13. □

Theorem 3.15. *Let $A(n)$ be the number of perfect powers in the open interval $((n-1)^2, n^2)$. We have*

$$A(n) \leq (2n - 2)$$

$$\lim_{n \to \infty} \frac{A(n)}{2n - 2} = 0$$

Proof. Clearly

$$A(n) \leq n^2 - (n-1)^2 - 1 = 2n - 2$$

On the other hand (Theorem 3.11)

$$\frac{A(n)}{2n-2} \leq \frac{F(n)}{2n-2}$$

□

Question: Is the sequence $A(n)$ bounded?

In the following theorem we prove that almost all the differences is not repeated.

Theorem 3.16. *Let us consider the set of the first n consecutive differences.*

$$d_1, d_2, \ldots, d_n$$

Let $B(n)$ be the number of distinct differences in this set. We have

$$\lim_{n\to\infty} \frac{B(n)}{n} = 1$$

Let $B_1(n)$ be the number of distinct differences in this set that appear only one time. We have

$$\lim_{n\to\infty} \frac{B_1(n)}{n} = 1$$

Let $B_2(n)$ be the number of distinct differences in this set that appear at least two times. We have

$$\lim_{n\to\infty} \frac{B_2(n)}{n} = 0$$

Proof. We have

$$1 + d_1 + d_2 + \cdots + d_n = P_{n+1}$$

Let us consider the equation $i^2 \leq P_{n+1}$. This equation has the following solutions

$$i = 1, 2, \ldots, \left\lfloor \sqrt{P_{n+1}} \right\rfloor$$

Consequently we have the following $\left\lfloor \sqrt{P_{n+1}} \right\rfloor$ open intervals.

$$\left(0, 1^2\right), \left(1^2, 2^2\right), \ldots, \left(\left(\left\lfloor \sqrt{P_{n+1}} \right\rfloor - 1\right)^2, \left\lfloor \sqrt{P_{n+1}} \right\rfloor^2\right)$$

and

$$P_{n+1} \in \left[\left\lfloor \sqrt{P_{n+1}} \right\rfloor^2, \left(\left\lfloor \sqrt{P_{n+1}} \right\rfloor + 1\right)^2\right)$$

Therefore (see Theorem 3.10) we have

$$\left\lfloor \sqrt{P_{n+1}} \right\rfloor - S\left(\left\lfloor \sqrt{P_{n+1}} \right\rfloor\right) - 1 \leq B(n) \leq n$$

That is,

$$\frac{\left\lfloor \sqrt{P_{n+1}} \right\rfloor}{n} - \frac{S\left(\left\lfloor \sqrt{P_{n+1}} \right\rfloor\right)}{\left\lfloor \sqrt{P_{n+1}} \right\rfloor} \frac{\left\lfloor \sqrt{P_{n+1}} \right\rfloor}{n} - \frac{1}{n} \leq \frac{B(n)}{n} \leq 1 \tag{3.26}$$

Now, $P_n \sim n^2$ (see equation (1.1)). Therefore $\sqrt{P_n} \sim n$. Besides $\frac{S(n)}{n} \to 0$ (see Theorem 3.10). Consequently the left side in equation (3.26) has limit 1. That is, we have

$$\lim_{n\to\infty} \frac{B(n)}{n} = 1 \tag{3.27}$$

Note that

$$B(n) = B_1(n) + B_2(n) \tag{3.28}$$

Suppose that $\frac{B_2(n)}{n}$ has not limit zero. Consequently there exist $\alpha > 0$ such that for infinite values of n we have

$$\frac{B_2(n)}{n} > \alpha \tag{3.29}$$

We have the inequality

$$n - B_1(n) - 2B_2(n) \geq 0$$

That is,

$$n \geq B_1(n) + 2B_2(n) \tag{3.30}$$

Equations (3.28) and (3.30) give

$$n \geq B(n) + B_2(n)$$

That is,

$$1 \geq \frac{B(n)}{n} + \frac{B_2(n)}{n} \tag{3.31}$$

There exists n_0 such that if $n \geq n_0$ we have (see equation (3.27))

$$\frac{B(n)}{n} > 1 - \frac{\alpha}{2} \tag{3.32}$$

Equations (3.29) and (3.32) give

$$\frac{B(n)}{n} + \frac{B_2(n)}{n} > 1 + \frac{\alpha}{2} \tag{3.33}$$

Equations (3.31) and (3.33) are an evident contradiction. Therefore we have

$$\lim_{n\to\infty} \frac{B_2(n)}{n} = 0 \tag{3.34}$$

Finally, equations (3.27), (3.28) and (3.34) give

$$\lim_{n\to\infty} \frac{B_1(n)}{n} = 1$$

□

Let us consider an open interval $(k^2, (k+1)^2)$ without perfect powers. We shall call this interval a A-interval. On the other hand, if the open interval $(k^2, (k+1)^2)$ has some perfect power then we shall call this interval a B-interval. A A-chain is a sequence of a finite number of consecutive A-intervals. The number of A-intervals in a A-chain we shall call the lenght of the A-chain. A B-chain is a sequence of a finite number of consecutive B-intervals. The number of B-intervals in a B-chain we shall call the lenght of the B-chain. In the extremes of a A-chain there are two B-chains and in the extremes of a B-chain there are two A-chains. Therefore the sequence of intervals

$$(k^2,(k+1)^2) \qquad (k=0,1,2,\dots)$$

becomes a sequence of A-chains and B-chains. In this sequence there are infinite B-chains and infinite A-chains (see Theorem 3.10 and Theorem 3.11).

Let us consider the set of the first n consecutive differences.

$$d_1, d_2, \dots, d_n$$

If $d_i < d_{i+1}$ we shall call this an increasing gap. The number of increasing gaps in the first n consecutive differences we denote $X(n)$. If $d_i > d_{i+1}$ we shall call this a decreasing gap. The number of decreasing gaps in the first n consecutive differences we denote $Y(n)$. If $d_i = d_{i+1}$ we shall call this a equal gap. The number of equal gaps in the first n consecutive differences we denote $Z(n)$. Clearly we have

$$X(n)+Y(n)+Z(n)=n-1 \tag{3.35}$$

In the following theorem we prove that the incresing gaps have density 1.

Theorem 3.17. *In the sequence of differences $d_1, d_2, d_3, \dots$ there are infinite increasing gaps. Besides*

$$X(n) \sim n$$

In the sequence of differences $d_1, d_2, d_3, \dots$ there are infinite decreasing gaps. Besides

$$Y(n)=o(n)$$

On the other hand, we also have

$$Z(n)=o(n)$$

Proof. We have

$$1+d_1+d_2+\cdots+d_n=P_{n+1}$$

Let us consider the equation $i^2 \le P_{n+1}$. This equation has the following solutions

$$i=1,2,\dots,\left\lfloor \sqrt{P_{n+1}} \right\rfloor$$

Consequently we have the following $\left\lfloor \sqrt{P_{n+1}} \right\rfloor$ open intervals.

$$\left(0,1^2\right),\left(1^2,2^2\right),\dots,\left(\left(\left\lfloor \sqrt{P_{n+1}} \right\rfloor -1\right)^2, \left\lfloor \sqrt{P_{n+1}} \right\rfloor^2\right) \tag{3.36}$$

and

$$P_{n+1} \in \left[\left\lfloor \sqrt{P_{n+1}} \right\rfloor^2, \left(\left\lfloor \sqrt{P_{n+1}} \right\rfloor +1\right)^2\right)$$

In the intervals (3.36) there are A-chains and B-chains, the last either A-chain or B-chain can be interrupted.

The number of A-intervals in the intervals (3.36) is (see Theorem 3.10)

$$\left\lfloor \sqrt{P_{n+1}} \right\rfloor - S\left(\left\lfloor \sqrt{P_{n+1}} \right\rfloor\right) = g(n)n \tag{3.37}$$

where $g(n) \to 1$. These A-intervals form the A-chains in the intervals (3.36) where the last A-chain can be interrupted. Note that each A-chain of lenght L

has $L-1$ increasing gaps and an interrupted A-chain with L' A-intervals has $L'-1$ increasing gaps. On the other hand, the number of B-intervals in the intervals (3.36) is

$$S\left(\left\lfloor\sqrt{P_{n+1}}\right\rfloor\right)=o(n) \tag{3.38}$$

These B-intervals form the B-chains in the intervals (3.36) where the last B-chain can be interrupted. Therefore the number of B-chains will be (see (3.38))

$$o(n) \tag{3.39}$$

Equations (3.37) and (3.39) give

$$g(n)n-o(n)\leq X(n)$$

Besides we have the trivial inequality

$$X(n)\leq n-1$$

Hence

$$X(n)\sim n$$

and (see equation (3.35))

$$Y(n)=o(n)$$

$$Z(n)=o(n)$$

Let us consider two consecutive intervals, the A-interval $(k^2,(k+1)^2)$ and the B-interval $((k+1)^2,(k+2)^2)$. Clearly there exists a decreasing gap. Consequently there exist infinite decreasing gap in the sequence of differences $d_1, d_2, d_3, \ldots$. $\square$

It is well-known that the series

$$\sum_{n=1}^{\infty}\frac{1}{P_n}$$

converges. This fact is also a direct consequence of the formula $P_n\sim n^2$ (see equation (1.1)).

Let us consider the series

$$\sum_{n=1}^{\infty}\frac{d_n}{P_n} \tag{3.40}$$

Note that

$$\lim_{n\to\infty}\frac{d_n}{P_n}=\lim_{n\to\infty}\frac{P_{n+1}-P_n}{P_n}=\lim_{n\to\infty}\left(\frac{P_{n+1}}{P_n}-1\right)=0$$

Since $P_{n+1}\sim P_n\sim n^2$ (see equation (1.1)).

In the following theorem we prove that the series (3.40) diverges.

Theorem 3.18. *The series (3.40) diverges. Besides*

$$\sum_{k=1}^{n}\frac{d_k}{P_k}<C\log n$$

where C is a positive constant.

Proof. We have $P_1 = 1^2$, $P_2 = 2^2$, $P_3 = 8 < 3^2$. Consequently if $n \geq 1$ then

$$P_n \leq n^2. \tag{3.41}$$

Since the squares are perfect powers.

On the other hand, it is well-known the following formula

$$\sum_{i=1}^{n} \frac{1}{i} = \log n + \gamma + g(n). \tag{3.42}$$

where $g(n) \to 0$ and γ is the Euler's constant.

Now, we have (Theorem 3.1, Corollary 3.2 and (3.41))

$$\begin{aligned}
\sum_{i=1}^{n} \frac{d_i}{P_i} &\geq \sum_{\substack{i \leq n \\ d_i > fi}} \frac{d_i}{P_i} \geq \sum_{\substack{i \leq n \\ d_i > fi}} \frac{fi}{i^2} = f \sum_{\substack{i \leq n \\ d_i > fi}} \frac{1}{i} \geq f \sum_{i=n-v(n)+1}^{n} \frac{1}{i} \\
&= f \sum_{i=w(n)+1}^{n} \frac{1}{i}
\end{aligned} \tag{3.43}$$

If the number of d_i such that $d_i \leq fi$ is $N \geq 0$ then $w(n) = N$ from a certain value of n and consequently

$$f \sum_{i=w(n)+1}^{n} \frac{1}{i} = f \sum_{i=N+1}^{n} \frac{1}{i} \to \infty \tag{3.44}$$

Therefore (see (3.43) and (3.44))

$$\sum_{i=1}^{\infty} \frac{d_i}{P_i} = \infty$$

On the other hand, if $w(n) \to \infty$ then we have (equation (3.42) and Corollary 3.2)

$$\begin{aligned}
f \sum_{i=w(n)+1}^{n} \frac{1}{i} &= f \left(\sum_{i=1}^{n} \frac{1}{i} - \sum_{i=1}^{w(n)} \frac{1}{i} \right) = f \left(\log n + g(n) - \log w(n) - g(w(n)) \right) \\
&= f \left(\log n + g(n) - \log \left(n \frac{w(n)}{n} \right) - g(w(n)) \right) \\
&= -f \log \left(\frac{w(n)}{n} \right) + h(n) \to \infty
\end{aligned} \tag{3.45}$$

where $h(n) \to 0$.

Therefore (see (3.43) and (3.45))

$$\sum_{i=1}^{\infty} \frac{d_i}{P_i} = \infty$$

Hence, we have proved that the series (3.40) diverges.

There exist n_0 such that if $k \geq n_0$ we have

$$P_k > (1 - \epsilon) k^2$$

Since $P_k \sim k^2$. Therefore we have (see Lemma 1.2)

$$\begin{aligned}\sum_{k=1}^{n} \frac{d_k}{P_k} &= \sum_{k=1}^{n_0-1} \frac{d_k}{P_k} + \sum_{k=n_0}^{n} \frac{d_k}{P_k} \leq \sum_{k=1}^{n_0-1} \frac{d_k}{P_k} + \sum_{k=n_0}^{n} \frac{2k}{(1-\epsilon)k^2} = \sum_{k=1}^{n_0-1} \frac{d_k}{P_k} \\ &+ \frac{2}{1-\epsilon} \sum_{k=n_0}^{n} \frac{1}{k} < C \log n\end{aligned}$$

□

Theorem 3.19. *If $0 < \alpha < 1$ then the series*

$$\sum_{k=1}^{\infty} \frac{d_k^{\alpha}}{P_k} \tag{3.46}$$

converges.

Consequently $\alpha = 1$ is the least real positive number such that the series (3.46) diverges (Theorem 3.18).

Proof. There exist n_0 such that if $k \geq n_0$ we have

$$P_k > (1-\epsilon)k^2$$

Since $P_k \sim k^2$. Therefore we have (see Lemma 1.2)

$$\begin{aligned}\sum_{k=1}^{n} \frac{d_k^{\alpha}}{P_k} &= \sum_{k=1}^{n_0-1} \frac{d_k^{\alpha}}{P_k} + \sum_{k=n_0}^{n} \frac{d_k^{\alpha}}{P_k} \leq \sum_{k=1}^{n_0-1} \frac{d_k^{\alpha}}{P_k} + \sum_{k=n_0}^{n} \frac{(2k)^{\alpha}}{(1-\epsilon)k^2} = \sum_{k=1}^{n_0-1} \frac{d_k^{\alpha}}{P_k} \\ &+ \frac{2^{\alpha}}{1-\epsilon} \sum_{k=n_0}^{n} \frac{1}{k^{2-\alpha}}\end{aligned}$$

where $(0 < \alpha < 1)$. □

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