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CORE OF A CONVEX MIXTURE SET AND CONVEX FUNCTIONS

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Abstract. We broach the notions of face, core and convex function of a convex mixture set. A version of famous Karush-Kuhn-Tucker theorem is proved. Several examples to illustrate are also given.

1. Introduction

In the early fifties of last century, the study of abstract convexity was initiated with the focus for an axiom system that defines a convex set. In some way it generalizes the classical concept of a Euclidean convex set. von Neuman and Mongernstern [15] in their classic book "Theory of Games and Economic Behavior" used utility theory to model economics problems. The expected utility representation was stated using an algebra of combining of compound lotteries. Stone [14] and Herstein and Milnor [9] tried to give axiomatic mathematical postulates by introducing mixture sets. Afterward Fishburn et al. [4, 5, 6] and Kelly and Weiss [10] further developed it at a high level of abstraction. The algebra of mixture sets and mixture preserving functions provide a mathematical frame work in which the basic question of utility theory can be modelled and application in linear utility functions and decision theoretic notions of lottery sets are developed. Gudder et al. [7, 8] in their excellent expository articles on convexity and mixtures presented applications to behavioral, social and physical sciences. An advantage of the abstract approach in mathematical economics and decision theory is that it focuses ones attention on the axioms of the system. Even though these mathematical tools are a fundamental part of the treatment of the subject, the researchers in decision theory have not shown much interest to further investigation in mixture sets and their applications. However, Mongin [11]; note on mixture sets in decision theory gave new wings to this theory. In more recent works Rommeswinkel [13] and Galaabaatar et al. [3] have started to investigate and obtained several important results by merging preferences with mixture structure. In this paper we further continue to study face and core of a convex mixture set and convex functions.

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This paper is set up as follows: A brief overview of mixture set is presented in Section 2. Section 3, broaches the notions of face and core of a convex mixture set. In Section 4, we introduce and study properties of convex functions. In Section 5, applications to constrained optimization problems are shown and a version of famous Karush-Kuhn-Tucker Theorem is derived. Conclusion is given in Section 6.

2. Preliminaries

First, we present some notation pertaining to mixture set and related works that will be used afterward.

Definition 2.1. [3, Definition 1] A set X is called a mixture set if for any $x, y \in X$ and for all real number α we can associate another element $x\alpha y$, which is again in X, and for all $x, y \in X$ and all α, β;
i. $x1y = x$, ii. $x\alpha y = y(1-\alpha)x$, iii. $(x\alpha y)\beta y = x(\alpha\beta)y$.

Remark 2.2. Definition 2.1 further implies $(x\alpha y)\beta(x\gamma y) = x(\alpha\beta + (1-\beta)\gamma)y$.

Definition 2.3. A nonempty subset M of a mixture set X is said to be convex if for all $x, y \in M$ and $\alpha \in [0,1]$, we have $x\alpha y \in M$.

Definition 2.4. [11] A function h from a mixture set M to a real vector space Z is called mixture preserving (MP) if $h(x\alpha y) = \alpha h(x) + (1-\alpha)h(y)$.

Definition 2.5. A mixture set X is said to have property ($\mathtt{C}$) if for all $x, y, z \in X$ it satisfies;
(C_1) For any $\alpha \in (0,1)$, $x\alpha y = x\alpha z$ implies $y = z$.
(C_2) For any $\alpha, \beta \in [0,1]$ with $\alpha\beta \neq 1$,

$$(x\alpha y)\beta z = x(\alpha\beta)(y\frac{\beta(1-\alpha)}{1-\alpha\beta}z). \tag{2.1}$$

The following theorem expresses' the essential features of the results of Stone [14] and Mongin [11] and is presented for use later on.

Theorem 2.6. [11] *For any mixture set X, the following are equivalent;*
i. X has property ($\mathtt{C}$) .
ii. X is isomorphic to a convex subset Y of some real linear space Z.

Let $x, y \in X$, we shall be using the following notation for subsets of X;

$$\begin{aligned}
[x,y] &= \{u : u = x\alpha y \text{ for some } \alpha \in [0,1]\},\\
(x,y] &= \{u : u = x\alpha y \text{ for some } \alpha \in [0,1)\},\\
[x,y) &= \{u : u = x\alpha y \text{ for some } \alpha \in (0,1]\},\\
(x,y) &= \{u : u = x\alpha y \text{ for some } \alpha \in (0,1)\}.
\end{aligned}$$

Throughout rest of this work we assume that X is a mixture set with property ($\mathtt{C}$) unless stated otherwise.

Theorem 2.7. [1, Article 29] *Helly's theorem: Let M be a finite family of convex sets in R^n such that, for $k \leq n+1$, any k members of M have a nonempty intersection. Then the intersection of all members of M is nonempty.*

3. Core

Definition 3.1. A face of a nonempty convex set M of a mixture set X is a nonempty set $F \subset M$ with the property that if $x, y \in M$, $\alpha \in [0,1]$, and $x\alpha y \in F$ then $x, y \in F$. A nonempty face $F \neq M$ is called a proper face.

Next we study various properties of a face of a nonempty convex subset of a mixture set.

Example 3.2. Any convex set M itself is its own face. The empty set ϕ is a face of any convex set M.

Remark 3.3. A face F is a subset of M so that any $[y, z] \subset M$, with $(x, y) \cap F \neq \phi$ must lie in F.

A way proper faces are constructed is via functions. Let $h : X \to Y \subset Z$ and $g : Y \to \mathbb{R}.$, where h be a MP function and g be a linear function. The following theorem provides sufficient conditions for construction of a proper face.

Theorem 3.4. *Let $goh : X \to \mathbb{R}$ be a non constant function satisfying*

$$\sup_{x \in X} goh(x) = \gamma < \infty. \tag{3.1}$$

Then the set $\{x : goh(x) = \gamma\} = F$ (if nonempty) is a proper face of X.

Proof. Assume without loss of generality that F is nonempty. First we show that F is a convex set . Let $x, y \in F$ i.e. $goh(x) = \gamma$ and $goh(y) = \gamma$. Now for any $\alpha \in [0,1]$,

$$\begin{aligned} goh(x\alpha y) &= g(h(x\alpha y)) = g(\alpha h(x) + (1-\alpha)h(y)) \\ &= \alpha g(h(x)) + (1-\alpha)g(h(y)) = \alpha goh(x) + (1-\alpha)goh(y) = \gamma. \end{aligned}$$

Thus F is a convex set. Let x, y be any arbitrary points in X with $x\alpha y \in F$. In case $\alpha = 0$, Definition 2.1(i & ii) implies $y \in F$ and if $\alpha = 1$, Definition 2.1(i) implies $x \in F$. When $x, y \in X, \alpha \in (0,1)$ and $x\alpha y \in F$, then $goh(x\alpha y) = \gamma$ and $goh(x) \leq \gamma$, $goh(y) \leq \gamma$ imply $goh(x) = \gamma$, $goh(y) = \gamma$. Hence $x, y \in F$. Since goh is not constant therefore F is a proper subset of X. □

The set $\{x : goh(x) = \gamma\}$ is also called exposed set. If it is singleton then that point is called exposed point. Next, we show that a face of a proper face of a convex set is a face of the convex set.

Theorem 3.5. *Let F be a proper face of a convex subset M of a mixture set X. A subset P of F is a face of $F \Leftrightarrow$ it is a face of M.*

Proof. Let P be a face of M, $x \in P$ and $x \in (y, z) \subset F$. Then $x \in (y, z) \subset M$. Therefore $y, z \in M$. Hence $y, z \in P$.. It further implies that P is a face of F.

Conversely,if P is a face of F, $x \in P$ and $x \in [y, z] \subset M$. As x is in F and F is a face of M, it imply that $y, z \in F$. Therefore $[y, z] \subset F$. As.P is a face of F and $y, z \in P$. Hence P is a face of M. □

Definition 3.6. Let M be a convex subset of a mixture set X. Core of M $core(M)$ is define as

$$core(M) = \{x \in M : \forall y \in M \ \exists \ z \in M \text{ such that } x \in (y, z)\}. \tag{3.2}$$

We next show that core of any convex set is convex.

Theorem 3.7. *Let M be a convex subset of a mixture set X then $core(M)$ is convex.*

Proof. Let x, y be two distinct points in $core(M)$ and $z \in (x, y)$ Now for any $a \in M$ there exists $b, c \in M$ such that $x \in (a, b)$ and $y \in (a, c)$. Now for α, β, γ in $(0, 1)$ we have

$$x = b\alpha a, \quad y = c\beta a, \quad z = y\gamma x.$$

Using Definition 2.1(ii) and Property (C_2), it further implies that

$$\begin{aligned} z &= (b\alpha a)\gamma(c\beta a) = (b(\frac{\alpha(1-\gamma)}{\alpha(1-\gamma)+\beta\gamma})c)(\alpha(1-\gamma)+\beta\gamma)a \\ &= e(\alpha(1-\gamma)+\beta\gamma)a, \end{aligned}$$

where $e = b(\frac{\alpha(1-\gamma)}{\alpha(1-\gamma)+\beta\gamma})c \in M$ by convexity of M. Therefore $z \in (a, e)$ with e in M. Thus $z \in core(M)$. Hence $core(M)$ is convex. □

Example 3.8. Let X be a mixture set of eventually zero sequences with $x\alpha y = \alpha x + (1-\alpha)y$ for $\alpha \in [0, 1]$.Let

$$M = \{x \in X : x_i \in [0, 1]\}.$$

Obviously M is convex. Let $x = (x_1, x_2, x_3, ..., x_\alpha, ..) \in M$ then there exists $\beta \in \mathbb{N}$ such that $x_\alpha = 0$ for all $\alpha \geqslant \beta$. Next choose $y = (y_1, y_2, y_3, ..., y_\alpha, ..) \in M$ such that y has similar entry with x up till $\beta - 1$ entry, has 1 in the $\beta - th$ entry and zero after $\beta - th$ entry. Let $z \in M$ be such that $x \in (y, z)$ then there exists γ in $(0, 1)$ such that $x = \gamma y + (1-\gamma)z$. It implies that $0 = x_\beta = \gamma y_\beta + (1-\gamma)z_\beta = \gamma + (1-\gamma)z_\beta$.It further implies that $z_\beta = \frac{-\gamma}{1-\gamma} < 0$. Hence a contradiction. Therefore $Core(M) = \phi$.

Remark 3.9. Since it is possible that $core(M) = \phi$ (Example 3.8) therefore $E \subset F$ did not imply $core(E) \subset core(F)$.

Lemma 3.10. *Let $w, x, y, z \in X$.If $w \in (y, z)$, then for all $u \in (w, x)$ there exists $v \in (z, x)$ such that $u \in (y, v)$.*

Proof. If $w \in (y, z)$, and $u \in (w, x)$ then there are $\alpha, \beta \in (0, 1)$ such that $w = y\alpha z$, $u = w\beta x$. Property (C_2) implies that,

$$u = w\beta x = (y\alpha z)\beta x = y(\alpha\beta)(z(\frac{\beta(1-\alpha)}{1-\alpha\beta})x) = y(\alpha\beta)v, \tag{3.3}$$

where $v = z(\frac{\beta(1-\alpha)}{1-\alpha\beta})x \in (z, x)$. Equality 3.3 implies that $u \in (y, v)$. □

Proposition 3.11. *Let M be a convex subset of a mixture set X. If $x \in core(M)$ and $y \in M$ then $[x, y) \subset core(M)$*

Proof. Let $x \in core(M)$ and $y \in M$. Choose an arbitrary point $z \in (x, y)$. Using Definition 3.6 of core, for any $a \in M$ there exists $b \in M$ such that $x \in (a, b)$. Lemma 3.10 further implies that there exists $c \in (b, y)$ such that $z \in (a, c)$. As M is convex and $b, y \in M$. Therefore $c \in M$. Using Definition 3.6 it further implies that $z \in core(M)$. Hence $[x, y) \subseteq core(M)$. □

The following two theorems illustrate that core of core of M is equal to core of M.

Theorem 3.12. *Let M be a convex subset of a mixture set X. If $x \in core(M)$ and $y \in M$ then there exists $e \in core(M)$ such that $x \in (e, y) \subseteq core(M)$.*

Proof. Let $x \in core(M)$ and $y \in M$ then there must be some $e \in M$ such that $x \in (e, y)$. Now using Proposition 3.11 we obtain $[x, e) \subset core(M)$. Thus $x \in (e, y) \subseteq core(M)$. □

Theorem 3.13. *Let M be a convex subset of a mixture set X then $core(core(M)) = cor(M)$.*

Proof. Let $x \in core(M)$. Now for any $y \in core(M)$ we also have $y \in M$. Using Theorem 3.12 there must be $e \in core(M)$ such that $x \in (e, y) \subseteq core(M)$. Therefore $x \in core(core(M))$.Thus $core(M) \subseteq core(core(M))$. *Converse* is obvious from Definition 3.6. Hence $core(core(M)) = core(M)$. □

4. Convex functions

A significant role is played by convex functions in several areas of mathematics. Convex functions play an important role in the study of optimization problems (see [2, 12]). In this section we study convex functions on a mixture set.

Definition 4.1. A real valued function f on a mixture set X is said to be convex if $f(x\lambda y)) \leq \lambda f(x) + (1 - \lambda) f(y)$ for all x, y in X and $\lambda \in [0, 1]$.

In case, $f(x\lambda y)) < \lambda f(x) + (1-\lambda) f(y)$ the function f is called strictly convex.

Example 4.2. Let $\Im$ be the family of all closed subintervals of $[0, 1]$ and $I_i = [a_i, b_i], I_j = [a_j, b_j] \in \Im$, define $I_i \lambda I_j = [\lambda a_i + (1 - \lambda) a_j, \lambda b_i + (1 - \lambda) b_j]$ Then Lebesgue measure is a convex function on mixture set $\Im$.

Example 4.3. Let $\Im$ be the family of all closed balls of $\mathbb{R}^2$ and $B_1[(\eta_1, \eta_2), r_1]$, $B_2[(\theta_1, \theta_2), r_2] \in \Im$. Define

$$\begin{aligned} B_1 \alpha B_2 &= (B_1[(\eta_1, \eta_2), r_1]) \alpha (B_2[(\theta_1, \theta_2), r_2]) \\ &= B[(\alpha\eta_1 + (1 - \alpha)\theta_1, \alpha\eta_2 + (1 - \alpha)\theta_2), \alpha r_1 + (1 - \alpha) r_2]. \end{aligned}$$

Now the function $f : \Im \to \mathbb{R}$ given by $f(B[(\eta, \gamma), r] = \sqrt{(\eta^2 + \gamma^2)} + r$ is convex.

Remark 4.4. In case X is a convex subset of a linear space, $\alpha = \frac{1}{2}$, and $x \frac{1}{2} y = \frac{x+y}{2}$ then f is convex if and only f is Jensen convex.

Proposition 4.5. *Let X be a convex mixture set and $f : X \to \mathbb{R}$ be any convex function Then*

i. Any restriction g of f to a convex subset M of X, is a convex function.
ii. Function σf for $\sigma \in [0, 1]$ is a convex function.

Proof. i. Since M is convex, therefore restriction g of f to a convex subset M of X is well defined. Convexity of g on M follows from the convexity of f on X.

ii. Using Definition 4.1

$$\begin{aligned}\sigma f(x\lambda y) &\leq \sigma[\lambda f(x) + (1-\lambda) f(y)] \\ &= \lambda\sigma f(x) + (1-\lambda)\sigma f(y) \leq \lambda\sigma f(x) + (1-\lambda\sigma) f(y).\end{aligned}$$

Hence σf is a convex function. □

Theorem 4.6. *Let X be a convex mixture set and $f, g : X \to \mathbb{R}$ be two convex functions Then the function $h = \max\{f, g\}$ is convex.*

Proof. Using Definition 4.1, for any x, y in X and $\lambda \in [0, 1]$,

$$f(x\lambda y) \leq \lambda f(x) + (1-\lambda) f(y) = \lambda h(x) + (1-\lambda) h(y),$$

and

$$g(x\lambda y) \leq \lambda g(x) + (1-\lambda) g(y) = \lambda h(x) + (1-\lambda) h(y).$$

Thus $h(x\lambda y) = \max\{f(x\lambda y), g(x\lambda y)\} \leq \lambda h(x) + (1-\lambda) h(y)$. Hence h is a convex function. □

Proposition 4.7. *Let $f, g : X \to \mathbb{R}$ be two convex functions on a mixture set X and $h = \max\{f, g\}$. Then there exists $\alpha \in [0, 1]$ such that for $x \in X$,*

$$0 \leq h(x) \Rightarrow 0 \leq \alpha f(x) + (1-\alpha) g(x).$$

Proof. For any x in X, $0 \leq \max\{f(x), g(x)\}$ implies there exists $\alpha \in [0, 1]$ such that $0 \leq \alpha f(x) + (1-\alpha) g(x)$. □

Theorem 4.8. *Suppose that Y and Z are two convex subsets of X ,$Y \cap Z = M \neq \phi$ and $f, g : X \to \mathbb{R}$ are such that f is convex on Y and g is convex on Z. Let $P = \{x \in X : \max\{f(x), g(x)\} < \infty\}$. Then $M \cap P$ is a convex set and $h = \max\{f, g\}$ is a convex function on $M \cap P$.*

Proof. Let x, y be any two points in $M \cap P$ then $x, y \in Y \cap Z$ and $max\{f(x), g(x)\} < \infty$, $max\{f(y), g(y)\} < \infty$. For any fix $\gamma \in [0, 1]$, convexity of $Y \cap Z$ implies that $x\gamma y \in Y \cap Z = M$.Therefore M is convex set. Next we show that P is a convex set. From convexity of f and g, we obtain for any $x, y \in P$, $\lambda \in [0, 1]$,

$$f(x\lambda y) \leq \lambda f(x) + (1-\lambda) f(y) = \lambda h(x) + (1-\lambda) h(y) < \infty,$$

and

$$g(x\lambda y) \leq \lambda g(x) + (1-\lambda) g(y) = \lambda h(x) + (1-\lambda) h(y) < \infty.$$

Thus $x\lambda y \in P$.Thus P is a convex set. It further implies that $M \cap P$ is a convex set. Now for the function h defined on $M \cap P$,we have

$$\begin{aligned}\max\{f(x\lambda y), g(x\lambda y)\} &\leq \lambda \max\{f, g\}(x) + (1-\lambda)\max\{f, g\}(y) \\ &= \lambda h(x) + (1-\lambda) h(y) < \infty.\end{aligned}$$

Hence h is a convex function on $M \cap P$. □

5. Applications

In this section as an application we prove existence of global minimizer of a convex function and existence of a solution for a constrained optimization problem.

Proposition 5.1. *Any real valued strictly convex function on a convex mixture set X has at most one global minimizer.*

Proof. Let $f : X \to \mathbb{R}$ be a convex function. Assume that there are two distinct points x, y in X such that

$$f(x) = f(y) = \inf_{x\in X} f(x).$$

Using convexity of mixture set X, we obtain $x\frac{1}{2}y$ is again in X and strict convexity of f further implies that $f(x\frac{1}{2}y) < \frac{1}{2}f(x) + \frac{1}{2}f(y) = \inf_{x\in X} f(x)$. Thus a contradiction. Hence $x = y$. □

Theorem 5.2. *Let X be a convex mixture set and $f, g : X \to \mathbb{R}$ be two convex functions. Then there exists $\lambda \in [0, 1]$ such that*

$$0 \leq \max\{\lambda f(x) + (1-\lambda)f(y), \lambda g(x) + (1-\lambda)g(y)\}$$

if and only if

$$0 \leq \alpha f(x) + (1-\alpha)g(x) \quad \alpha \in [0, 1]$$

Proof. $\Rightarrow$: Assume that

$$0 \leq \max\{\lambda f(x) + (1-\lambda)f(y), \lambda g(x) + (1-\lambda)g(y)\}.$$

For any x in X define $I_x = \{\alpha : \ 0 \leq \alpha f(x) + (1-\alpha)g(x)\} \subset [0, 1]$. Now the inequality $0 \leq \alpha f(x) + (1-\alpha)g(x)$ is equivalent to $\underset{x\in X}{\cap} I_x \neq \phi$. Obviously each I_x is a compact convex subset of $[0, 1]$. Using Helly's theorem 2.7 (for $n = 1$), $\underset{x\in X}{\cap} I_x \neq \phi$ $\Leftrightarrow$ $I_x \cap I_y \neq \phi$ for any two distinct x, y in X. To show that $I_x \cap I_y \neq \phi$, let x and y be any two fixed arbitrary points in X, then by hypothesis there exists $\lambda \in [0, 1]$ such that

$$0 \leq 0 \leq \max\{\lambda f(x) + (1-\lambda)f(y), \lambda g(x) + (1-\lambda)g(y)\}.$$

Proposition 4.7 further implies that there exists $\alpha \in [0, 1]$ such that

$$\begin{aligned} 0 &\leq \alpha[\lambda f(x) + (1-\lambda)f(y)] + (1-\alpha)[\lambda g(x) + (1-\lambda)g(y)] \\ &= \lambda[\alpha f(x) + (1-\alpha)g(x)] + (1-\lambda[\alpha f(y) + (1-\alpha)g(y)] \end{aligned}$$

If $\lambda = 0$, we obtain $0 \leq \alpha f(y) + (1-\alpha)g(y)$ and in case $\lambda = 1$, we have $\alpha f(x) + (1-\alpha)g(x)$. It implies that $\alpha \in \ I_x \cap I_y$. Therefore $\underset{x\in X}{\cap} I_x \neq \phi$. Hence $0 \leq \alpha f(x) + (1-\alpha)g(x)$.

$\Leftarrow$: Let $0 \leq \alpha f(x) + (1-\alpha)g(x)$ for some $\alpha \in [0,1]$. Let $x, y \in X$ and $\lambda \in [0,1]$.Now $0 \leq \alpha f(x) + (1-\alpha)g(x)$ implies

$$\begin{aligned} 0 &\leq \lambda[\alpha f(x) + (1-\alpha)g(x)] + (1-\lambda)[\alpha f(y) + (1-\alpha)g(y)] \\ &= \alpha[\lambda f(x) + (1-\lambda)f(y)] + (1-\alpha)[\lambda g(x) + (1-\lambda)g(y)] \\ &\leq \max\{\lambda f(x) + (1-\lambda)f(y), \lambda g(x) + (1-\lambda)g(y)\}. \end{aligned}$$

Hence the result. □

Lemma 5.3. *Let f, g be two real valued convex functions on a convex mixture set X, satisfying*

$$0 \leq \max_{x \in X}\{f(x), g(x)\},$$

then there exists $\alpha \in [0,1]$ such that $0 \leq \alpha f(x) + (1-\alpha)g(x)$ holds.

Proof. Let $x, y \in X$ and $\alpha \in [0,1]$, then $x\alpha y \in X$. Using convexity of functions f, g we further have

$$0 \leq \max\{f(x\alpha y), g(x\alpha y)\} \leq \max\{\alpha f(x) + (1-\alpha)f(y), \alpha g(x) + (1-\alpha)g(y)\}.$$

Theorem 5.2 further implies that $0 \leq \alpha f(x) + (1-\alpha)g(x)$. □

Now we prove a version of famous Karush-Kuhn-Tucker Theorem.

Theorem 5.4. *Let X be a convex mixture set and $f, g : X \to \mathbb{R}$ be two convex functions. Assume that $f(x_0) = 0$ and x_0 is a solution of constrained optimization problem*

$$Minimize \quad f(x) \quad subject\ to \quad g(x) \leq 0. \tag{5.1}$$

Then there exist $\alpha \in [0,1]$ such that $\alpha g(x_0) = 0$ and for x in X

$$0 \leq (1-\alpha)f(x) + \alpha g(x). \tag{5.2}$$

Proof. Let x_0 be a solution of the above optimization problem, then for any x in X,

$$f(x) < f(x_0) = 0 \quad \text{and} \quad g(x) \leq 0$$

can not hold simultaneously. Thus

$$0 \leq \max_{x \in X}\{f(x), g(x)\}.$$

Lemma 5.3 further implies that there exist $\alpha \in [0,1]$ such that inequality (5.2) holds. As x_0 is a solution to optimization problem (5.1), that is , we have $g(x_0) \leq 0$. Therefore

$$0 \leq (1-\alpha)f(x_0) + \alpha g(x_0) = \alpha g(x_0) \leq 0. \tag{5.3}$$

Since $g(x_0) \leq 0$ therefore the only way that 5.3 holds is that $g(x_0) = 0$. It further implies that for $\alpha \in [0,1]$, $\alpha g(x_0) = 0$. □

Theorem 5.5. *Let X be a convex mixture set and $f, g : X \to \mathbb{R}$ be two convex functions. Suppose that there exist a point $x_0 \in X$ and $\alpha \in [0,1]$ such that $\alpha g(x_0) = 0$ and for x in X*

$$0 \leq (1-\alpha)f(x) + \alpha g(x), \tag{5.4}$$

then x_0 is a solution of constrained optimization problem

$$Minimize \quad f(x) \quad subject\ to \ \ g(x) \leq 0. \tag{5.5}$$

Proof. Let $x \in X$ be an admissible point with respect to minimization problem; $Minimize \quad f(x)$ subject to $g(x) \leq 0$. Now from hypothesis, we have

$$\begin{aligned} (1-\alpha)f(x_0) &= (1-\alpha)f(x_0) + \alpha g(x_0) = 0 \\ &\leq (1-\alpha)f(x) + \alpha g(x) \leq (1-\alpha)f(x). \end{aligned}$$

Since $\alpha \in [0,1]$ it further implies that $f(x_0) \leq f(x)$. Hence x_0 is a minimal point of f. □

Remark 5.6. In fact Theorem 5.5 is a converse of Theorem 5.4.

6. Conclusion

In this article, I introduced the definitions and studied properties of face, core and convex functions. In addition to applications, these concepts are important them self, as they generalize the notion of convexity. Future work is concerned with the development of Minkowski–Krein–Milman property, some type of separation of convex sets and more general maximum theorem. Applications of these results in gambling, economics, game theory, multi criteria decision problems, social choice theory, optimization theory and approximation theory will be an interesting research problem.

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