

A PROBLEM WITH PERIODIC BOUNDARY CONDITIONS FOR THE NON-FOURIER HEAT EQUATION

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ABSTRACT. All diffusion equations are based on the infinite velocity of potential fields, which leads to well-known paradoxes. Consequently, in non-stationary processes, the evolution of these quantities do not completely obey the above equations due to the lack of parameters in them that take into account the finite rate of potential growth.

In the heat conduction theory, numerous generalizations of the Fourier law are used as a remedy for these issues. The article gives a brief overview of generalizations of the Fourier law. Some mathematical issues of well-posed boundary value problems for the Guyer-Krumhansl model are discussed. As an application, a boundary value problem for a general quasilinear equation with periodic boundary conditions is considered. Schauder-type a priori estimates are established and the uniqueness of the solution is proved.

1. INTRODUCTION

Some recent observations have indicated that the way heat behaves in materials with varying compositions can deviate from the traditional Fourier law. This deviation results in a different thermal diffusivity and necessitates the determination of extra thermal properties, as mentioned in [40]. The interest in several aspects of heat transfer under nonequilibrium conditions in nanoscale systems is primarily driven by technological requirements. This includes applications like managing heat in microelectronics and rapidly processing promising metamaterials using ultrafast lasers [1, 39].

Furthermore, when heat transfer occurs on extremely small spatial and temporal scales, it gives rise to unconventional non-Fourier effects. These effects involve temperature fluctuations that depend on size and distance, changes in thermal conductivity, and abrupt shifts in boundary temperatures. These phenomena have spurred extensive research and generated a substantial body of literature addressing various theoretical aspects of the process, [39]-[36].

The hyperbolic heat equation (HHCE) is the most straightforward and widely recognized extension of the traditional parabolic Fourier heat equation. It incorporates elements of both diffusion and wave-like heat transfer mechanisms,

Date: Received: Aug 19, 2023; Accepted: Oct 8, 2023.

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2020 *Mathematics Subject Classification.* Primary 46L55; Secondary 44B20.

Key words and phrases. PDEs, applied mathematics, non-Fourier model, a priori estimates, existence and uniqueness .

resulting in a scenario where temperature experiences abrupt changes at a constant rate, without any gradual smoothing [37]-[19].

Wave-like energy propagation has been observed in metals when they are exposed to brief laser irradiation, as reported in [41]. Additionally, there has been a recent discovery of a phenomenon referred to as "second sound" in graphite at temperatures exceeding 100K, as noted in [9]. Moreover, researchers have applied the hyperbolic transfer equation to investigate the swift solidification of alloys under circumstances significantly distant from local equilibrium within the context of mass diffusion [32].

However, HHCE is not suitable for describing thermal conductivity in nanoscale systems. In these systems, non-local spatial effects, cf. [44, 43, 34], and the ballistic aspect of heat transfer, cf. [34, 16], can give rise to non-Fourier phenomena. These include size-dependent thermal behavior, variations in conductivity, and abrupt changes in boundary temperatures [10].

Nonlocal spatial influences [37] have been extensively examined in various domains. These include the realm of thermal conductivity within nanoscale systems [34, 10, 23], as well as in metals exposed to ultra-brief laser irradiation, spanning picoseconds and femtoseconds [39, 38, 31, 27, 45]. Furthermore, nonlocal spatial effects have garnered attention in contexts such as biological systems, cf. [7], and diverse heterogeneous systems [14]-[15].

Many experiments conducted at room temperature have aimed to validate the existence of Maxwell-Cattaneo-Vernot (MCV) behavior, cf. [2]. This MCV equation is employed to model the phenomenon known as "second sound," which represents the dissipative form of heat wave propagation, [21, 8]. Despite numerous experimental efforts, the suitability of the MCV equation for describing behavior at room temperature remains unconfirmed. It is worth noting that in addition to MCV, there exist several other extensions of the Fourier equation, including the Guyer-Krumhansl (GK) equation, [8]-[13], and their various modifications, [29].

There has recently been an increased interest in studying the Guyer-Krumhansl equation, both analytically and numerically [47, 13, 33]. Additionally, several other theoretical investigations are underway, aiming to enhance our understanding and detect observable phenomena in nanomaterials.

Henceforth, we only consider the GK equation, which possesses a simple Fourier law-like structure and can be applied to problems at room temperature.

Work of [42] reports the outcomes of experiments involving a thermal pulse conducted on different artificial and natural materials. These experiments were conducted using large-scale samples at room temperature. The findings indicate that the temperature evolution deviates from the predictions of the Fourier law but can be effectively described using the Guyer-Krumhansl equation.

In the experiment of [42], the internal energy balance equation is

$$\rho c T_t + q_x = 0, \quad (1.1)$$

where c is the specific heat, T is the temperature, and their product $e_{sp} = cT$ is the specific internal energy. Furthermore, ρ is the density and q is the heat flux in the direction of heat propagation. The evolutionary heat flow equation has the

form

$$\tau q_t + q + kT_x - l^2 q_{xx} = 0, \quad (1.2)$$

where k is the Fourier thermal conductivity, τ is the relaxation time, and l^2 is a non-negative material parameter of the GK equation and is expressed using the characteristic length scale l .

If the heat flow is eliminated in (1.1)-(1.2) and the system is written in terms of temperature, we get

$$T_t + \tau T_{tt} = \alpha (T_{xx} + bT_{txx}), \quad (1.3)$$

where $\alpha = k/\rho c$ is the thermal diffusivity, $b = l^2/\tau\alpha$ is the coefficient characterizing the deviation from the Fourier thermal conductivity. It can be seen that the solutions of the Fourier equation are the solutions of (1.3) for $b = 1$, i.e., $\alpha = l^2/\tau$.

Henceforth, this formula will be referred to as the Fourier resonance condition. If $b < 1$, then the solutions (1.1)-(1.2) have wave-like characteristics, and if $b > 1$, then the solutions are superdiffusive.

According to experts, mathematical models of locally nonequilibrium transport do not always agree with each other. This indicates the absence of a single mathematical apparatus of research. Consequently, the models themselves require final construction, detailed mathematical justification, and experimental confirmation [39, 11, 19].

Therefore, the purpose of the work is to begin the development of some elements of the mathematical apparatus according to the theory of non-Fourier models. This article gives a brief overview of extensions of Fourier's law. Some mathematical issues of setting correct boundary value problems for the GK model are discussed. For a quasilinear relaxation equation of a special form, a boundary value problem with periodic boundary conditions is considered. A way of reducing the problem to a new problem for parabolic equations is shown. Next, a method is proposed for obtaining a priori estimates of the Schauder type and applying the Leray-Schauder method.

2. PRELIMINARY RESULTS

2.1. Initial-boundary conditions. The initial condition for T_t can be determined from the GK constitutive equation. On the other hand, the boundary conditions are defined through constitutive equations. Since the defining equation GK introduces additional temporal and spatial derivatives of the heat flux, the temperature gradient is no longer able to set boundary conditions such as heat flux. For Neumann-type boundary conditions, a q or mixed (without exclusion of variables) representation is recommended [13]. This is a key aspect of solving the GK equation.

2.2. T and q -representations. A "primary" field variable can be chosen based on the application. Particular choice can sometimes simplify the determination of boundary conditions. In the context of the GK equation, the potential candidates for this primary variable are either temperature T or heat flux q , leading to one of the following equations:

T-representation: $\tau T_{tt} + T_t - \alpha T_{xx} - l^2 T_{txx} = 0$,

q-representation: $\tau q T_{tt} + q_t - \alpha q_{xx} - l^2 q_{txx} = 0$.

Note that in the T -representation, the boundary condition for q would require the knowledge of q_{xx} , and thus is not available.

On the other hand, in the q -representation the boundary condition for T is meaningless.

In this paper, system (1.1)-(1.2) is used. Note that the solution of the Fourier equation can be computed from the GK model when $l^2/\tau q = \alpha$, which is called the Fourier resonance.

These coefficients indicate the existence of parallel heat exchange pathways, introducing two distinct time scales into the model. Interestingly, equation (2.1) bears a notable resemblance to the two-temperature Sobolev model [33]. In this model, it is assumed that there are two heat conduction channels, both following the Fourier law, and they are interconnected through balance equations. A similar representation can be derived for the average temperature, and while its outcome closely aligns with the GK equation, there are differences, particularly in terms of applicability, which may be restricted to situations where the material constituents are well-known. In the broader context of the GK equation, the coefficients τ , α , and l^2 should correspond to a specific temperature history.

2.3. On the solution algorithm. Let us rewrite (1.3) as

$$T_t + \tau T_{tt} = \alpha(T_{xx} + \tau T_{txx}). \quad (2.1)$$

For $\alpha > 0$, $\tau > 0$, introducing $u(t, x)$ via

$$T(t, x) + \tau T_t(t, x) = u(t, x), \quad (2.2)$$

a parabolic equation for $u(t, x)$ from (2.1) is deduced:

$$u_t(t, x) = \alpha u_{xx}(t, x). \quad (2.3)$$

If the initial and boundary conditions are provided for equation (2.1), then by utilizing equation (2.2), the required conditions for the function $u(t, x)$ can be recovered. To find $T(x, t)$, the following problem is considered

$$T(t, x) + \tau T_t(t, x) = u(t, x),$$

$$T(0, x) = T_0(x).$$

Its solution has the form (after finding $u(t, x)$)

$$T(t, x) = T_0(x)e^{-\frac{t}{\tau}} + \tau \int_0^t e^{-\frac{1}{\tau}(t-\eta)} u(\eta, x) d\eta.$$

3. NONLINEAR PERIODIC PROBLEM

3.1. Formulation of the problem. The method described above can be employed when investigating boundary value problems involving quasilinear relaxation equations with a specific form.

In a region

$$Q = [(t, x) : 0 < t < H, -l < x < l]$$

consider a problem with periodic boundary conditions

$$(T + \tau T_t)_t = a(t, x, T + \tau T_t, T_x + \tau T_{tx})(T_{xx} + \tau T_{txx}) + b(t, x, T + \tau T_t, T_x + \tau T_{tx}), \quad (3.1)$$

$$T(x, 0) = 0, T_t(x, 0) = 0, -l \leq x \leq l, \quad (3.2)$$

$$T(t, -l) = T(t, l), T_x(t, -l) = T_x(t, l), 0 < t < H. \quad (3.3)$$

Taking into account (2.2), from (3.1)-(3.3), we obtain the parabolic problem

$$u_t = a(t, x, u, u_x)u_{xx} + b(t, x, u, u_x), \quad (3.4)$$

$$u(0, x) = 0, \quad (3.5)$$

$$u(t, -l) = u(t, l), u_x(t, -l) = u_x(t, l). \quad (3.6)$$

We assume that the following basic conditions are met:

A) For $(t, x) \in Q$, $|u| \leq M$ for arbitrary p satisfying $a(t, x, u, p) \geq a_0 > 0$.

B) $\frac{|b(t, x, u, p)|}{a(t, x, u, p)} \leq K(p^2 + 1)$, $K = \text{const} > 0$.

We will adhere to the notation adopted in [17, 20], where Γ is the set-theoretic union of the points of the lower base and the sides of the rectangle Q .

$$Q_\delta^+ = \{(t, x) : 0 < \delta \leq t \leq T, \delta - l \leq x \leq l\},$$

$$Q_\delta^- = \{(t, x) : \delta \leq t \leq T, -l \leq x \leq l - \delta\},$$

$$Q_\delta^\delta = \{(t, x) : 0 \leq t \leq T, |x| \leq l - \delta\}, Q^\delta = Q_\delta^+ \cap Q_\delta^-.$$

In the problems considered below, the symmetry of the boundaries along the x direction is not essential.

For $u(t, x)$ defined in a region D and for $\forall \gamma \in (0, 1)$, we define

$$|u|_\gamma^D = \sup_D |u(t, x)| + \sup_{(t, x) \in D, (\tau, y) \in D} \frac{|u(t, x) - u(\tau, y)|}{(|t - \tau| + |x - y|^2)^{\gamma/2}}, |u|_{1+\gamma}^D = |u|_\gamma^D + |u_x|_\gamma^D, \\ |u|_{2+\gamma}^D = |u|_{1+\gamma}^D + |u_{xx}|_\gamma^D + |u_t|_\gamma^D.$$

3.2. A Priori estimates.

Theorem 3.1. *Let $u(t, x), u_x(t, x) \in C(Q)$ satisfy the conditions (3.5), (3.6) and the equation (3.4) inside Q , except perhaps at $x = 0$. Let for $(t, x) \in Q$, $|u| \leq M$ and arbitrary p , functions a and b satisfy conditions A), B), and*

$$a(t, -l, u, p) = a(t, l, u, p), \quad b(t, -l, u, p) = b(t, l, u, p). \quad (3.7)$$

Then for $(t, x) \in Q$,

$$|u_x(t, x)| \leq C(M, a_0, K) = M_1. \quad (3.8)$$

If $a_1 = \max a(t, x, u, p)$, $a_2 = \max |b(t, x, u, p)|$ in the area of $\{(t, x) \in Q, |u| \leq M, |p| \leq M_1\}$, then

$$|u(t_1, x) - u(t_2, x)| \leq C(M, a_1, b_2, K, M_1) |t_1 - t_2|^{1/2}. \quad (3.9)$$

Assume $u_{xx}, u_{tx} \in L_2(Q)$. Then there exists $\gamma = \gamma(M, a_1, b_2, K)$, such that

$$|u|_{1+\gamma}^Q = C(M, a_0, a_1, K), \quad 0 < \gamma < 1. \quad (3.10)$$

Proof. The bound $|u_x| \leq C$ for Q_0^δ is an immediate consequence of the results of [17], Theorem 2. It remains to establish the validity of the estimate up to the sides of the rectangle. In [17], taking advantage of $u|_{x=\pm l} = 0$, the author extends the solution through the boundaries in an odd way. We propose to extend the function $u(t, x)$ outside Q as follows:

$$u(t, x) = u(t, 2l + x) \text{ for } -3l \leq x \leq -l, \quad (3.11)$$

$$u(t, x) = u(t, x - 2l) \text{ for } l \leq x \leq 3l. \quad (3.12)$$

We assume that the coefficients of (3.4) are extended along x direction according to (3.11)-(3.12). A new function (let us keep the notation $u(t, x)$) at all points of the rectangles

$$R_\pm u(t, x) = \left\{ (t, x) : 0 \leq t \leq T, \left| x \pm \frac{3}{2}l \right| \leq \frac{3}{2}l \right\}$$

has a continuous derivative u_x and satisfies the "extended" equation of the form (3.4) (e.g., for $l < x < 3l$, $u_t = a(t, x - 2l, u, u_x)u_{xx} + b(t, x - 2l, u, u_x)$, with the same properties as in the hypotheses of theorem. Using the well-known "internal" results, we obtain an estimate for $|u_x|$ in rectangles, whose union contains Q . Since obtaining inner bounds is based on the maximum principle, the assertions of the theorem are completely preserved when the function $u(t, x)$ is continuous in Q has a continuous derivative u_x and satisfies the equation (3.4) in Q everywhere except at points of a finite number of lines $x = \text{const}$ (see Theorem 4.3 in [18]).

Owing to the above derived estimates for $|u|$ and $|u_x|$ in Q , by virtue of [18, Theorem 4.5], we have the bound (3.9).

Internal evaluation

$$|u|_{1+\gamma}^{Q^{2\delta}} \leq C(M, a_0, a_1, K), \quad (3.13)$$

follows from the results of [17], Theorem 3. By virtue of the obtained results, we can consider the function $u(t, x)$ as a solution to some linear equation

$$u_t = \bar{a}(t, x) u_{xx} + \bar{b}(t, x),$$

with bounded and Holder continuous coefficients. To obtain an estimate up to the boundary, as in the first statement of the theorem, we extend $u(t, x)$ according to the rule (3.9)-(3.12). Further (see the proof of Theorem 4 [17]), for the solution of the "extended" equation, internal a priori estimates of the form (3.13), in the rectangles enclosing the rectangle Q , the results of the work ([17] Theorem 3) on the Holder property of a generalized solution are applied. Therefore, we get the estimate (3.10). $\square$

Lemma 3.2. *The solution $u(t, x)$ of the problem (3.4)-(3.6) satisfies*

$$|u(t, x)| \leq \frac{b_1 \exp\{\beta T\}}{b - \beta} = M, \quad (3.14)$$

where $b > \beta$.

Proof. The proof proceeds as follows. Let $v = e^{-\beta t} u(t, x)$ and consider the equation for $v(t, x)$ at the interior points of positive maximum and negative minimum. Taking into account the conditions A) and B), using the theorem on the sign of

the derivative at the boundary point of the extremum, as well as estimating the modulus of the solution of the linear equation, the proof of the lemma is completed. $\square$

Theorem 3.3. *Let all the assumptions of Theorem 3.1 hold. Moreover, assume that $|b_x| \leq B_1$, and that there exists $K_1 > 0$ with $a(t, x, u, p)$ satisfying*

$$|a_x|, |a_u|, |a_p| \leq K_1.$$

A function $u(t, x)$ satisfying the conditions of the problem in Q is continuous in Q together with the derivatives u_t, u_x, u_{xx} , and inside Q along with u_{tx}, u_{xxx} . Suppose $|u|_{2+\gamma}^{Q_0^{2\delta}} < \infty$. Then

$$|u|_{2+\gamma}^{Q_0^{2\delta}} \leq C(M, a_0, a_1, K, \delta). \quad (3.15)$$

If, in addition, we have

$$a(t, -l, u, p) = a(t, l, u, p), \quad b(t, -l, u, p) = b(t, l, u, p) \quad (3.16)$$

and $|u|_{2+\gamma}^{Q_0^{2\delta}} < \infty$, then

$$|u|_{2+\gamma}^Q \leq C(M_1, a_0, a_1, K_1, K). \quad (3.17)$$

Proof. Taking the x derivative of the equation (3.4) inside Q , we obtain

$$p_t = ap_{xx} + \bar{b}(t, x, u, p, p_x), \quad (3.18)$$

for $p = u_x(t, x)$, where

$$\bar{b} = (ap_x + a_up + b_p)p_x + bu_p + (a_x + b_x).$$

Then the initial condition gives

$$p(0, x) = 0. \quad (3.19)$$

The second condition of (3.6) gives

$$p(t, -l) = p(t, l). \quad (3.20)$$

Then assuming $x = \pm l$ in the equation (3.4), taking into account (3.16) from the first equality in (3.6), we find

$$p_x(t, -l) = p_x(t, l). \quad (3.21)$$

Now applying Theorem 3.1 (formula (3.10)) (the functions p, a and $\bar{b}$ satisfy the conditions of this theorem) to the problem (3.18)-(3.21), we obtain

$$|u_x(t, x)|_{1+\gamma}^Q \leq C.$$

The estimate for u_t then directly follows from (3.4). $\square$

3.3. Existence and uniqueness of a solution. We first establish the following theorem pertaining to linear parabolic equations.

Theorem 3.4. *Let $u(t, x)$ solve*

$$Lu \equiv a(t, x)u_{xx} + b(t, x)u_x + e(t, x)u - u_t = f(t, x), \quad (t, x) \in Q, \quad (3.22)$$

and satisfy the conditions (3.5)- (3.6). If the coefficients a, b, e and $f(t, x)$ satisfy the Holder condition, then there exists a solution to the problem (3.22), (3.5), (3.6) with

$$|u|_{2+\alpha}^Q \leq C |f|_{\alpha}^Q. \quad (3.23)$$

Proof. Considering the results of [20], we have

$$|u|_{2+\alpha}^{Q_0^\delta} \leq C_0 |f|_{\alpha}^Q. \quad (3.24)$$

Therefore, it suffices for us to set the estimate (3.23) for $t > 0$. We reduce the equation (3.22) to a homogeneous form using the solution

$$u_0(t, x) = \int_0^t d\eta \int_{-l}^l \Gamma(t, x, \eta, \xi) d\xi, \quad (3.25)$$

where $\Gamma(t, x, \eta, \xi)$ is the fundamental solution of the equation (3.22). Using the substitution $v = u + u_0$, given that $Lu_0 = -f$, we obtain the problem

$$\begin{aligned} Lv &= 0, \\ v(t, -l) &= v(t, l) + r_1(t), \quad v_x(t, -l) = v_x(t, l) + r_2(t), \\ v(0, x) &= 0 \end{aligned} \quad (3.26)$$

where

$$r_1(t) = u_0(t, -l) - u_0(t, l), \quad r_2(t) = u_{0x}(t, -l) - u_{0x}(t, l).$$

Next, we refine the smoothness of the functions $r_i(t)$, $i = 1, 2$. From the parabolic potential theory [20, 5], it follows that if $f(t, x)$ satisfies the Holder condition, then u has continuous derivatives u_{0x} , u_{0xx} in Q and u_{0t} , and the bound

$$|u_0|_{2+\alpha}^Q \leq C_0 |f|_{\alpha}^Q$$

holds. The functions $r_i(t)$, $i = 1, 2$, depend on the values $u_0(t, l)$ ($u_0(t, -l)$) and $u_x(t, l)$ ($u_{0x}(t, -l)$). It is known that the function $\Gamma(t, x, \eta, \xi)$ for $x \neq \xi$, is a sufficiently smooth function. Hence $r_i(t)$, $i = 1, 2$, are sufficiently smooth functions as well. The solution of (3.26) is then sought as the sum of the parabolic potentials of a simple layer

$$v(t, x) = \int_0^t \Gamma(t, x; \eta, -l) \psi_1 d\eta + \int_0^t \Gamma(t, x; \eta, l) \psi_2(\eta) d\eta, \quad (3.27)$$

where $\psi_i(t)$, $i = 1, 2$, are unknown densities. It is known that for the potential of a simple layer

$$V(t, x) = \int_0^t \Gamma(t, x; l, \eta) \psi(\eta) d\eta$$

there is an estimate

$$|V|_{2+\alpha}^Q \leq C_0 |\psi|_{1+\alpha}. \quad (3.28)$$

Therefore, if we prove in (3.27) that $\psi_i(t) \in C^{1+\alpha}(0, T]$, then proof will be complete.

Implementing the boundary conditions of the (3.26) problem, we find

$$\begin{aligned} & \int_0^t \Gamma(t, -l; \eta, -l) \psi_1(\eta) d\eta + \int_0^t \Gamma(t, -l; \eta, l) \psi_2(\eta) d\eta \\ & - \int_0^t \Gamma(t, l; \eta, -l) \psi_1(\eta) d\eta - \int_0^t \Gamma(t, l; \eta, l) \psi_2(\eta) d\eta = r_1(t), \\ & -\frac{\sqrt{\pi}}{\sqrt{a(t, 0)}} \psi_1(t) + \int_0^t \Gamma_x(t, -l; \eta, -l) \psi_1(\eta) d\eta + \int_0^t \Gamma_x(t, -l; \eta, l) \psi_2(\eta) \\ & -\frac{\sqrt{\pi}}{\sqrt{a(t, 0)}} \psi_2(t) + \int_0^t \Gamma_x(t, l; \eta, -l) \psi_1(\eta) d\eta - \int_0^t \Gamma_x(t, l; \eta, l) \psi_2(\eta) d\eta = r_2(t). \end{aligned} \quad (3.29)$$

$$(3.30)$$

The equation (3.29) is reduced to an equation of the second kind

$$\begin{aligned} & \frac{d}{dt} \int_0^t \frac{r_1(\eta)}{\sqrt{t-\eta}} d\eta = r_1 \psi_1(t) + \int_0^t \frac{\partial}{\partial t} \left[\int_\eta^t \frac{\Gamma(z, -l; \eta, -l)}{\sqrt{t-z}} dz \right] \psi_1(\eta) d\eta - \pi \psi_2(t) - \\ & - \int_0^t \frac{\partial}{\partial t} \left[\int_\eta^t \frac{\Gamma(z, l; \eta, -l)}{\sqrt{t-z}} dz \right] \psi_2(\eta) d\eta + \int_0^t \frac{\partial}{\partial t} \left[\int_\eta^t \frac{\Gamma(z, -l; \eta, l)}{\sqrt{t-z}} dz \right] \psi_2(\eta) d\eta - \\ & - \int_0^t \frac{\partial}{\partial t} \left[\int_\eta^t \frac{\Gamma(z, l; \eta, l)}{\sqrt{t-z}} dz \right] \psi_1(\eta) d\eta. \end{aligned} \quad (3.31)$$

Making use of the properties of parabolic potentials, as well as integral inequalities, we get the estimates for solving the system in the form we need

$$|\psi_1|_{1+\alpha} \leq C_1 |f|_\alpha, \quad (3.32)$$

$$|\psi_2|_{1+\alpha} \leq C_2 |f|_\alpha. \quad (3.33)$$

Combining the obtained bounds, we get

$$|u|_{1+\alpha}^Q \leq C_2 |f|_\alpha^Q. \quad (3.34)$$

Thus, the theorem is proven. $\square$

Next, we shall establish the unique solvability of the problem (3.4)-(3.6) using the previously derived bounds and the Theorem 3.4.

Theorem 3.5. *Let the assumptions of Theorem 3.3 be satisfied and the matching condition $b(0, -l, 0, 0) = b(0, l, 0, 0)$ holds. Then there exists a solution to the problem (3.4)-(3.6) $u \in C^{2+\alpha}(Q)$ for some $\alpha \in (0, 1)$ and such a solution is unique.*

Proof. We will employ the Schauder fixed point principle. Let $\bar{C}^{1+\gamma}(Q)$ be the set of functions from $C^{1+\gamma}(Q)$ satisfying the conditions (3.5), (3.6). Consider the system

$$v_t = a(t, x, u, u_x) u_{xx} + b(t, x, u, u_x), \quad (3.35)$$

$$v(0, x) = 0, \quad (3.36)$$

$$v(t, -l) = v(t, l), \quad v_x(t, -l) = v_x(t, l), \quad (3.37)$$

where $u(t, x) \in \bar{C}^{1+\gamma}(Q)$.

The equation (3.35) is a linear equation with Holder continuous coefficients to which the Theorem 3.4 is applicable. Moreover, the uniqueness of the solution is a consequence of the extremum principle applicable to linear equations.

To demonstrate the possibility of solving a nonlinear problem, one can employ different theorems derived from the field of nonlinear equations, while taking into account the applicability of the uniqueness theorem for classical solutions. We will utilize the Leray-Schauder principle [5], along with the established a priori estimates $|\cdot|_{1+\alpha}$ for all potential solutions of nonlinear problems, in addition to the solvability theorem within Holder classes for linear problems.

In doing so, we use the results of the papers (Chapter VII, [5]; Chapter VI, [20]).

Problem (3.4)-(3.6) is considered together with a similar one-parameter family of problems

$$\begin{aligned} v_t &= a(t, x, \tau u, \tau u_x) v_{xx} + b(t, x, \tau u, \tau u_x), \\ v(0, x) &= 0, \\ v(t, -l) &= v(t, l), v_x(t, -l) = v_x(t, l), \end{aligned} \quad (3.38)$$

In the definition of the function $v(t, x)$, the function $u(t, x)$ is assumed to be given. Denote by $H^{1+\beta}$, $\beta \in (t, x)$, the Banach space of functions $u(t, x)$ on Q with the norm $|u| = |u|_{1+\beta}^Q$, which satisfy conditions (3.5), (3.6)

Under certain restrictions on the functions a and b , the problem (3.38) defines an operator F in $H^{1+\beta}$, which for each function $u \in H^{1+\beta}$ matches the solution v of the linear problem (3.38):

$$v = F(u; \tau). \quad (3.39)$$

The fixed points of this operator for $\tau = 1$ are solutions to the problem (3.4), (3.5), (3.6).

Let u^τ be one of the fixed points of the transformation F :

$u^\tau = F(u^\tau; \tau)$, i.e., u^τ is a solution to the equation

$$u_t = a(t, x, \tau u, \tau u_x) u_{xx} + b(t, x, \tau u, \tau u_x) \quad (3.40)$$

with boundary conditions (3.5), (3.6).

The equation (3.40) has the property that if for the equation (3.4) the conditions of one or another theorem from the previous paragraphs, in which the norm estimate is obtained, are valid, then the same conditions are valid for (3.40) at $0 \leq \tau \leq 1$.

So we can assume that the uniform boundedness in the norm $|u|_{2+\beta}^Q$ of all fixed points of the transformation $F(u, \tau)$ has been established. Now let's prove that (3.39) satisfies the conditions of the Leray-Schauder principle. The a priori estimates established above guarantee that the norms u^τ are uniformly bounded in the space in which the transformation $F(u, \tau)$ is considered.

Let us now show that $F(u, \tau)$ is uniformly continuous in u . Take two close elements u_1 and u_2 from $H^{1+1/3}$ and their corresponding $v_1 = F(u_1, \tau)$, $v_2 = F(u_2, \tau)$.

We have

$$\begin{aligned} (v_1 - v_2)_t &= a(t, x, \tau u_1, \tau u_{1x}) u_{xx} - a(t, x, \tau u_1, \tau u_{1x}) u_{2xx} \\ &\quad + u_{2xx} [a(t, x, \tau u_1, \tau u_{1x}) - a(t, x, \tau u_2, \tau u_{2x})] \\ &\quad + b(t, x, \tau u_1, \tau u_{1x}) - b(t, x, \tau u_2, \tau u_{2x}). \end{aligned}$$

Hence for $xv(t, x) = v_1 - v_2$ we find

$$v_t = \tilde{a}(t, x) u_{xx} + f(t, x), \quad (3.41)$$

where

$$\begin{aligned} \tilde{a}(t, x) &= a(t, x, \tau u_1, \tau u_{1x}), \\ f(t, x) &= v_{2xx}(t, x) [a(t, x, \tau u_1, \tau u_{1x}) - a(t, x, \tau u_2, \tau u_{2x})] \\ &\quad + b(t, x, \tau u_1, \tau u_{1x}) - b(t, x, \tau u_2, \tau u_{2x}). \end{aligned}$$

The solution of the equation (3.41) satisfies the boundary conditions of the problem (3.38).

Based on the results for linear equations, we have

$$|u|_{2+\beta}^Q \leq N_0 |f|_{\beta}^Q.$$

If we rewrite $f(t, x)$ as

$$\begin{aligned} f(t, x) &= (u_1 - u_2) \tau \left[u_{2xx} \int_0^1 a_u(\dots) d\eta + \int_0^1 b_u(\dots) d\eta \right] + \\ &\quad + (u_{1x} - u_{2x}) \tau \left[u_{2xx} \int_0^1 a_p(\dots) d\eta + \int_0^1 b_p(\dots) d\eta \right], \end{aligned}$$

then one can find that

$$|f|_{\beta}^Q \leq N_1 |u_1 - u_2|_{1+\beta}^Q.$$

Consequently,

$$|v|_{1+\beta}^Q \leq N_2 |u_1 - u_2|_{1+\beta}^Q.$$

The uniform continuity is proved similarly $F(u; \tau)$ as a function of τ . Let us now prove the complete continuity of the operator $F(u; \tau)$. For $u(t, x)$ with $|u|_{1+\beta}^Q \leq C$ and $\tau \in [0, 1]$ functions $v = F(u; \tau)$, as solutions to the (3.38) problem, have uniformly bounded norms $|v|_{2+\beta}^Q \leq C$.

The fact that the (3.38) problem has a unique solution for $\tau = 0$ follows from the maximum principle.

Thus, for every $\tau \in [0, 1]$ there is at least one fixed point $u^\tau(t, x)$ for $F(u; \tau)$, which will be a solution to the problem (3.4), (3.5), (3.6) from $C_{2+\beta}$. This concludes the proof of the Theorem. $\square$

4. CONCLUSION

In this work, we studied some questions related to the GK equation. In particular, we discussed possible boundary conditions for different GK models. Then we investigated a quasilinear relaxation equation of special form within a periodic setting. After reducing it to a new parabolic problem, we then obtained a priori estimates and established well-posedness using Leray-Schauder principle.

REFERENCES

- [1] Cahill, D. G., & others. (2003). Nanoscale thermal transport. *Journal of Applied Physics*, 93, 793–818. <https://doi.org/10.1063/1.1524305>
- [2] Cattaneo, C. (1958). Sur une forme de l'équation de la chaleur éliminant le paradoxe d'une propagation instantanée. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences*, 247, 431–433. [https://doi.org/10.1016/S0001-8708\(58\)80044-7](https://doi.org/10.1016/S0001-8708(58)80044-7)
- [3] El-Nabulsi, R. A. (2018). Nonlocal approach to nonequilibrium thermodynamics and non-local heat diffusion processes. *Continuum Mechanics and Thermodynamics*, 30, 889–915. <https://doi.org/10.1007/s00161-017-0587-1>
- [4] Fehér, A., & Kovács, R. (2021). On the evaluation of non-Fourier effects in heat pulse experiments. *International Journal of Engineering Science*, 169, 103577. <https://doi.org/10.1016/j.ijengsci.2021.103577>
- [5] Friedman, A. (2008). *Partial Differential Equations of Parabolic Type*. Courier Dover Publications.
- [6] Gandolfi, M., Giannetti, C., & Banfi, F. (2020). Temperonic crystal: A superlattice for temperature waves in graphene. *Physical Review Letters*, 125, 265901. <https://doi.org/10.1103/PhysRevLett.125.265901>
- [7] Gula, I. A., & Samsonov, A. V. (2015). A model for the expression of *gap* genes based on the Jeffreys-type equation. *Bioinformatics*, 31(5), 714–719. doi: 10.1093/bioinformatics/btu699
- [8] Guyer, R. A., & Krumhansl, J. A. (1966). Solution of the Linearized Phonon Boltzmann Equation. *Physical Review*, 148, 766–778. <https://doi.org/10.1103/PhysRev.148.766>
- [9] Huberman, S., & others. (2019). Observation of second sound in graphite at temperatures above 100 K. *Science*, 364, 375–379. <https://doi.org/10.1126/science.aau6524>
- [10] Joseph, M. A., & Cao, B. (2022). Electron Heat Source Driven Heat Transport in GaN at Nanoscale: Electron–Phonon Monte Carlo Simulations and a Two Temperature Model. *Materials*, 15(5), 1651. doi: 10.3390/ma15051651
- [11] Joseph, D. D., & Preziosi, L. (1989). Heat waves. *Reviews of Modern Physics*, 61, 41–73. <https://doi.org/10.1103/RevModPhys.61.41>
- [12] Kamynin, L. I. (1963). The method of thermal potentials for a parabolic equation with discontinuous coefficients. *Soviet Mathematics. Doklady*, 4(5), 1071–1105.
- [13] Kovács, R. (2018). Analytic solution of Guyer-Krumhansl equation for laser flash experiments. *International Journal of Heat and Mass Transfer*, 127, 631–636. <https://doi.org/10.1016/j.ijheatmasstransfer.2018.07.062>
- [14] Kovács, R. (2022). Mathematical aspects of non-Fourier heat equations. *Journal of Computational and Applied Mechanics*, 17, 1–12. doi: 10.26353/jerm.590
- [15] Kovács, R., Feher, A., & Sobolev, S. L. (2022). On the Two-Temperature Description of Heterogeneous Materials. *International Journal of Heat and Mass Transfer*, 194, 123021. doi: 10.1016/j.ijheatmasstransfer.2022.123021
- [16] Kovács, K. (2022). Analytical treatment of nonhomogeneous initial states for non-Fourier heat equations. *International Communications in Heat and Mass Transfer*, 134, 106021. <https://doi.org/10.1016/j.icheatmasstransfer.2021.106021>
- [17] Kruzhkov, S. N. (1967). Nonlinear parabolic equations with two independent variables. *Transactions of the Moscow Mathematical Society*, 16, 329–346.
- [18] Kruzhkov, S. N. (1979). Quasilinear parabolic equations and systems with two independent variables. *Proceedings of the Seminar I.G. Petrovsky*, Issue 5, 217–272.
- [19] Kudinov, V. A., Eremin, A. V., & Kudinov, I. V. (2017). The development and investigation of a strongly non-equilibrium model of heat transfer in fluid with allowance for the spatial and temporal non-locality and energy dissipation. *Thermophysics and Aeromechanics*, 24(6), 901–907. doi: 10.1134/S0869864317060023
- [20] Ladyzhenskaja, O. A., Solonnikov, V. A., & Ural'ceva, N. N. (1968). *Linear and Quasilinear Equations of Parabolic Type*. American Mathematical Society: Translations of Mathematical Monographs, Vol. 23, Providence, RI.

- [21] Landau, L. (1947). On the theory of superfluidity of Helium II. *Journal of Physics*, 11, 91–92. <https://doi.org/10.1088/0959-5309/11/1/306>
- [22] Li, H.-L., Hua, Y.-C., & Cao, B.-Y. (2018). A hybrid phonon Monte Carlo-diffusion method for ballistic-diffusive heat conduction in nano- and micro-structures. *International Journal of Heat and Mass Transfer*, 127, 1014–1022. <https://doi.org/10.1016/j.ijheatmasstransfer.2018.08.045>
- [23] Li, C., Tian, X., & He, T. (2020). Size-dependent buckling analysis of Euler-Bernoulli nanobeam under non-uniform concentration. *Archive of Applied Mechanics*, 90, 1845–1860. <https://doi.org/10.1007/s00419-020-01700-2>
- [24] Li, H.-L., Shiomi, J., & Cao, B.Y. (2020). Ballistic-Diffusive Heat Conduction in Thin Films by Phonon Monte Carlo Method: Gray Medium Approximation Versus Phonon Dispersion. *Journal of Heat Transfer*, 142, 112502. <https://doi.org/10.1115/1.4046616>
- [25] Lurie, S. A., & Belov, P. A. (2020). On the nature of the relaxation time, the Maxwell–Cattaneo and Fourier law in the thermodynamics of a continuous medium, and the scale effects in thermal conductivity. *Continuum Mechanics and Thermodynamics*, 32, 709–728. <https://doi.org/10.1007/s00161-019-00816-4>
- [26] Mazza, G., & others. (2021). Thermal dynamics and electronic temperature waves in layered correlated materials. *Nature Communications*, 12, 6904–6911. <https://doi.org/10.1038/s41467-021-27349-3>
- [27] Mikheeva, G. V., & Pashin, A. V. (2021). Investigation of heat transfer in metal nanofilms irradiated with ultrashort laser pulses: Two-temperature model. *Journal of Physics: Conference Series*, 2094, 22023. <https://doi.org/10.1088/1742-6596/2094/2/022023>
- [28] Picandet, V., & Challamel, N. (2021). Nonlocal thermal diffusion in one-dimensional periodic lattice. *International Journal of Heat and Mass Transfer*, 180, 121753. <https://doi.org/10.1016/j.ijheatmasstransfer.2021.121753>
- [29] Rogolino, P., Kovács, R., Ván, P., & Cimmelli, V. A. (2018). Generalized heat-transport equations: Parabolic and hyperbolic models. *Continuum Mechanics and Thermodynamics*, 30(6), 1245–1258. doi: 10.1007/s00161-018-0707-1
- [30] Serdyukov, S. I. (2018). Macroscopic Entropy of Non-Equilibrium Systems and Postulates of Extended Thermodynamics: Application to Transport Phenomena and Chemical Reactions in Nanoparticles. *Entropy*, 20, 802–819. <https://doi.org/10.3390/e20110802>
- [31] Sobolev, S. L. (1993). Two-temperature discrete model for nonlocal heat conduction. *Journal of Physics III France*, 3, 2261–2298. <https://doi.org/10.1051/jp3:1993225>
- [32] Sobolev, S. L. (2015). Rapid phase transformation under local non-equilibrium diffusion conditions. *Materials Science and Technology*, 31, 1607–1617. <https://doi.org/10.1179/1743284714Y.0000000722>
- [33] Sobolev, S. L. (2016). Nonlocal two-temperature model: Application to heat-transporting metals irradiated by ultrashort laser pulses. *International Journal of Heat and Mass Transfer*, 94, 138–144. <https://doi.org/10.1016/j.ijheatmasstransfer.2015.11.067>
- [34] Sobolev, S. L., Cao, B. Y., & Kudinov, I. V. (2017). Non-Fourier heat transport across 1D nano film between thermal reservoirs with different boundary resistances. *Physica E*, 128, 114610. <https://doi.org/10.1016/j.physe.2020.114610>
- [35] Sobolev, S. L. (2018a). On hyperbolic heat-mass transfer equation. *International Journal of Heat and Mass Transfer*, 122, 629–630. <https://doi.org/10.1016/j.ijheatmasstransfer.2017.12.011>
- [36] Sobolev, S. L. (2018b). Hyperbolic heat conduction, effective temperature, and third law for nonequilibrium systems with heat flux. *Physical Review E*, 97, 022122. <https://doi.org/10.1103/PhysRevE.97.022122>
- [37] Sobolev, S. L. (2018c). On hyperbolic heat-mass transfer equation. *International Journal of Heat and Mass Transfer*, 122, 629–630. <https://doi.org/10.1016/j.ijheatmasstransfer.2017.12.011>

- [38] Sobolev, S. L. (2018d). Hyperbolic heat conduction, effective temperature, and third law for nonequilibrium systems with heat flux. *Physical Review E*, 97, 022122. <https://doi.org/10.1103/PhysRevE.97.022122>
- [39] Sobolev, S. L. (2021). Local Nonequilibrium Electron Transport in Metals after Femtosecond Laser Pulses: A Multi-Temperature Hyperbolic Model. *Nanoscale Microscale Thermophys. Eng.*, 25, 153–165. [doi:10.1080/15567265.2020.1869314](https://doi.org/10.1080/15567265.2020.1869314)
- [40] Sobolev, S. L. (2022). Heat Transport on Ultrashort Time and Space Scales in Nano-sized Systems: Diffusive or Wave-like? *Materials*, 15, 4287. <https://doi.org/10.3390/ma15124287>
- [41] Suarez, C., Bron, W. E., & Juhasz, T. (1995). Dynamics and Transport of Electronic Carriers in Thin Gold Films. *Physical Review Letters*, 75, 4536–4539. <https://doi.org/10.1103/PhysRevLett.75.4536>
- [42] Ván, P., & others. (2017). Guyer-Krumhansl-type heat conduction at room temperature. *EPL (Europhysics Letters)*, 118, 50005. <https://doi.org/10.1209/0295-5075/118/50005>
- [43] Xu, M. (2018). A non-local constitutive model for nano-scale heat conduction. *International Journal of Thermal Sciences*, 134, 594–600. doi.org/10.1016/j.ijthermalsci.2018.07.029
- [44] Xu, M. (2020). Nonlocal heat conduction in silicon nanowires and carbon nanotubes. *Heat and Mass Transfer*, 57(3), 843–852. <https://doi.org/10.1007/s00231-020-02740-3>
- [45] Xue, T., Zhang, X., & Tamma, K. K. (2019). On a generalized non-local two-temperature heat transfer DAE modeling/simulation methodology for metal-nonmetal thermal interfacial problems. *International Journal of Heat and Mass Transfer*, 138, 508–515. <https://doi.org/10.1016/j.ijheatmasstransfer.2019.05.119>
- [46] Zhang, Y., & others. (2020). Coherent modulation of the electron temperature and electron-phonon couplings in a 2D material. *Proceedings of the National Academy of Sciences of the United States of America*, 117(17), 8788–8793. doi: 10.1073/pnas.1919205117
- [47] Zhukovsky, K. V. (2016). Exact solution of Guyer–Krumhansl type heat equation by operational method. *International Journal of Heat and Mass Transfer*, 96, 132–144. doi: 10.1016/j.ijheatmasstransfer.2015.12.028
- [48] Zhukovsky, K. V. (2016). Operational Approach and Solutions of Hyperbolic Heat Conduction Equations. *Axioms*, 5(3), 28. doi: 10.3390/axioms5030028

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