

AN ALTERNATIVE PROOF OF MIYASHITA'S THEOREM IN A SKEW POLYNOMIAL RING II

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ABSTRACT. Y. Miyashita gave characterizations of a separable polynomial and a Hirata separable polynomial in skew polynomial rings. In the previous paper, the author and S. Ikehata gave direct and elementary proofs of Miyashita's theorems in skew polynomial rings of Automorphism type $B[X; \rho]$ and Derivation type $B[X; D]$, respectively. The purpose of this paper is to give proofs for them in the general skew polynomial ring $B[X; \rho, D]$.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, all rings have an identity 1. Let A/B be a ring extension with common identity 1. A/B is called *separable* if the A - A -homomorphism of $A \otimes_B A$ onto A defined by $a \otimes b \mapsto ab$ splits, and A/B is called *Hirata separable* if $A \otimes_B A$ is A - A -isomorphic to a direct summand of a finite direct sum of copies of A . It is well known that a Hirata separable extension is separable. We denote

$$(A \otimes_B A)^A = \{\mu \in A \otimes_B A \mid z\mu = \mu z \text{ for any } z \in A\},$$

$$V_A(B) = \{g \in A \mid \alpha g = g\alpha \text{ for any } \alpha \in B\}.$$

Concerning separable extensions and Hirata separable extensions, the followings are well known.

Lemma 1.1. ([1, Definition 2]) *A/B is separable if and only if there exists $\sum_i z_i \otimes w_i \in (A \otimes_B A)^A$ such that $\sum_i z_i w_i = 1$.*

Lemma 1.2. ([13, Proposition 1]) *A/B is Hirata separable if and only if there exists $g_i \in V_A(B)$ and $\sum_j z_{ij} \otimes w_{ij} \in (A \otimes_B A)^A$ such that $1 \otimes 1 = \sum_i g_i \sum_j z_{ij} \otimes w_{ij} = \sum_i \sum_j z_{ij} \otimes w_{ij} g_i$.*

Let B a ring with identity element 1, ρ an automorphism of B , and D a ρ -derivation (i.e. an additive endomorphism of B such that $D(\alpha\beta) = D(\alpha)\beta + \rho(\alpha)D(\beta)$ for any $\alpha, \beta \in B$). By $B[X; \rho, D]$, we denote the skew polynomial ring in which the multiplication is given by $\alpha X = X\rho(\alpha) + D(\alpha)$ for any $\alpha \in B$. In particular, we write $B[X; \rho] = B[X; \rho, 0]$ and $B[X; D] = B[X; 1, D]$. Moreover, by $B[X; \rho, D]_{(0)}$, we denote the set of all monic polynomials f in $B[X; \rho, D]$ such

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that $fB[X; \rho, D] = B[X; \rho, D]f$. For any polynomial $f \in B[X; \rho, D]_{(0)}$, the residue ring $B[X; \rho, D]/fB[X; \rho, D]$ is a free ring extension of B . A polynomial f in $B[X; \rho, D]_{(0)}$ is called *separable* (resp. *Hirata separable*) in $B[X; \rho, D]$ if $B[X; \rho, D]/fB[X; \rho, D]$ is a separable (resp. Hirata separable) extension of B .

Throughout this article, let $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ be in $B[X; \rho, D]_{(0)}$ and

$$\begin{aligned} Y_0 &= X^{m-1} + X^{m-2}a_{m-1} + \cdots + Xa_2 + a_1, \\ Y_1 &= X^{m-2} + X^{m-3}a_{m-1} + \cdots + Xa_3 + a_2, \\ &\dots\dots \\ Y_j &= X^{m-j-1} + X^{m-j-2}a_{m-1} + \cdots + Xa_{j+2} + a_{j+1}, \\ &\dots\dots \\ Y_{m-2} &= X + a_{m-1}, \\ Y_{m-1} &= 1. \end{aligned}$$

In addition, we shall use the following conventions:

$$\begin{aligned} A &= B[X; \rho, D]/fB[X; \rho, D] \\ x &= X + fB[X; \rho, D] \in A \\ y_j &= Y_j + fB[X; \rho, D] \quad (0 \leq j \leq m-1) \\ B^\rho &= \{\alpha \in B \mid \rho(\alpha) = \alpha\} \\ B^D &= \{\alpha \in B \mid D(\alpha) = 0\} \\ B^{\rho, D} &= B^\rho \cap B^D \\ V_0 &= V_A(B) = \{g \in A \mid \alpha g = g\alpha \text{ for any } \alpha \in B\} \\ V_{m-1} &= \{h \in A \mid \rho^{m-1}(\alpha)h = h\alpha \text{ for any } \alpha \in B\} \\ (A \otimes_B A)^A &= \{\mu \in A \otimes_B A \mid z\mu = \mu z \text{ for any } z \in A\} \end{aligned}$$

In [9], Y. Miyashita studied separable polynomials and Hirata separable polynomials in skew polynomial rings. He proved the followings.

Proposition 1.3. ([9, Theorem 1.8]) *Let $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ be in $B[X; \rho, D]_{(0)}$. Then f is separable in $B[X; \rho, D]$ if and only if there exists $h \in V_{m-1}$ such that $\sum_{j=0}^{m-1} y_j h x^j = 1$.*

Proposition 1.4. ([9, Theorem 1.9]) *Let $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ be in $B[X; \rho, D]_{(0)}$. Then f is Hirata separable in $B[X; \rho, D]$ if and only if there exist $g_i \in V_0$ and $h_i \in V_{m-1}$ such that $\sum_i g_i x^{m-1} h_i = 1$ and $\sum_i g_i x^k h_i = 0$ ($0 \leq k \leq m-2$).*

Y. Miyashita proved the proposition above by making use of the theory of (*)-positively filtered rings. However, it seems not easy for one to comprehend his proofs. In the previous paper [15], the author and S. Ikehata gave direct and elementary proofs of the above propositions in $B[X; \rho]$ and $B[X; D]$, respectively. In this paper, we shall generalize our proofs for the skew polynomial ring $B[X; \rho, D]$.

Now we shall mention briefly some properties of the coefficients of polynomials in $B[X; \rho, D]_{(0)}$. The following lemma can be proved by a direct computation.

Lemma 1.5. ([2, Lemma 1.1]) *Let $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ be in $B[X; \rho, D]$. Then f is in $B[X; \rho, D]_{(0)}$ if and only if $\alpha f = f\rho^m(\alpha)$ for any $\alpha \in B$ and $Xf = f(X - \rho(a_{m-1}) + a_{m-1})$.*

In virtue of Lemma 1.5, we obtain the following.

Lemma 1.6. *Assume that $\rho D = D\rho$ and $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ is in $B[X; \rho, D]$. Then f is in $B[X; \rho, D]_{(0)}$ if and only if*

- (1) $a_i \rho^m(\alpha) = \sum_{j=i}^m \binom{j}{i} \rho^j D^{j-i}(\alpha) a_j \quad (\alpha \in B, 0 \leq i \leq m-1, a_m = 1)$
- (2) $D(a_i) = a_{i-1} - \rho(a_{i-1}) - a_i(\rho(a_{m-1}) - a_{m-1}) \quad (1 \leq i \leq m-1)$
- (3) $D(a_0) = a_0(\rho(a_{m-1}) - a_{m-1})$

Proof. Let α be an arbitrary element in B . By Lemma 1.5, f is in $B[X; \rho, D]_{(0)}$ if and only if $\alpha f = f\rho^m(\alpha)$ and $Xf = f(X - \rho(a_{m-1}) + a_{m-1})$. It is easy to see that $\alpha X^j = \sum_{i=0}^j \binom{j}{i} X^i \rho^j D^{j-i}(\alpha) a_j$ ($j \geq 0$) by an induction. Hence we obtain

$$\begin{aligned} \sum_{j=0}^m \alpha X^j a_j &= \sum_{j=0}^m \left(\sum_{i=0}^j \binom{j}{i} X^i \rho^j D^{j-i}(\alpha) \right) a_j \\ &= \sum_{i=0}^m X^i \left(\sum_{j=i}^m \binom{j}{i} \rho^j D^{j-i}(\alpha) a_j \right). \end{aligned}$$

This means that $\alpha f = f\rho^m(\alpha)$ implies (1), and conversely. By a direct computation, we have

$$\begin{aligned} &f(X - \rho(a_{m-1}) + a_{m-1}) \\ &= \sum_{i=0}^{m-1} X^i a_i X - \sum_{i=0}^{m-1} X^i a_i (\rho(a_{m-1}) - a_{m-1}) \\ &= \sum_{i=0}^{m-1} X^i (X\rho(a_i) + D(a_i)) - \sum_{i=0}^{m-1} X^i a_i (\rho(a_{m-1}) - a_{m-1}) \\ &= \sum_{i=0}^{m-1} X^{i+1} \rho(a_i) + \sum_{i=0}^{m-1} X^i (D(a_i) - a_i(\rho(a_{m-1}) - a_{m-1})) \\ &= X^m a_{m-1} + \sum_{i=1}^{m-1} X^i (\rho(a_{i-1}) + D(a_i) - a_i(\rho(a_{m-1}) - a_{m-1})) \\ &\quad + D(a_0) - a_0(\rho(a_{m-1}) - a_{m-1}). \end{aligned}$$

This means that $Xf = f(X - \rho(a_{m-1}) + a_{m-1})$ implies (2) and (3), and conversely. This completes the proof. $\square$

Corollary 1.7. *Assume that $\rho D = D\rho$ and $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ is in $B[X; \rho, D]_{(0)} \cap B^\rho[X]$. Then f is in $C(B^{\rho, D})[X]$, where $C(B^{\rho, D})$ is the center of $B^{\rho, D}$. Moreover,*

$$\alpha a_i = \sum_{j=i}^m (-1)^{j-i} \binom{j}{i} a_j \rho^{m-j} D^{j-i}(\alpha) \quad (\alpha \in B, 0 \leq j \leq m, a_m = 1)$$

Proof. Let $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ be in $B[X; \rho, D]_{(0)} \cap B^\rho[X]$. Then $a_i \in B^\rho$ ($0 \leq i \leq m-1$) implies $a_i \in B^D$ ($0 \leq i \leq m-1$) by Lemma 1.6 (2) and (3). Therefore $a_i \in C(B^{\rho, D})$ ($0 \leq i \leq m-1$) by Lemma 1.6 (1). Let α be an arbitrary element in B . By an easy induction, we see that $X^j \alpha = \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} \rho^{-j} D^{j-i}(\alpha) X^i$ ($j \geq 0$). Hence we obtain

$$\begin{aligned} \sum_{j=0}^m a_j X^j \rho^m(\alpha) &= \sum_{j=0}^m a_j \left(\sum_{i=0}^j (-1)^{j-i} \binom{j}{i} \rho^{m-j} D^{j-i}(\alpha) X^i \right) \\ &= \sum_{i=0}^m \left(\sum_{j=i}^m (-1)^{j-i} \binom{j}{i} a_j \rho^{m-j} D^{j-i}(\alpha) \right) X^i \end{aligned}$$

Therefore $\alpha f = f \rho^m(\alpha)$ implies our assertion. This completes the proof. $\square$

2. MAIN RESULT

The conventions and notations employed in the preceding section will be used in this section. We assume that $\rho D = D\rho$ and $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ is in $B[X; \rho, D]_{(0)} \cap B^\rho[X]$. Note that f is in $C(B^{\rho, D})[X]$ by Corollary 1.7. First we shall prove the following.

Lemma 2.1.

$$(A \otimes_B A)^A = \left\{ \sum_{j=0}^{m-1} y_j h \otimes x^j \mid h \in V_{m-1} \right\}.$$

Proof. Since $\{1, x, x^2, \dots, x^{m-1}\}$ is a free B -basis for A , every element in $A \otimes_B A$ has a form $\sum_{j=0}^{m-1} z_j \otimes x^j$ for some $z_j \in A$. Let $\sum_{j=0}^{m-1} z_j \otimes x^j$ be in $(A \otimes_B A)^A$ and α an arbitrary element in B . Since $\alpha \left(\sum_{j=0}^{m-1} z_j \otimes x^j \right) = \left(\sum_{j=0}^{m-1} z_j \otimes x^j \right) \alpha$ and $x^j \alpha = \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} \rho^{-j} D^{j-i}(\alpha) x^i$ ($j \geq 0$), we have

$$\begin{aligned} \sum_{j=0}^{m-1} \alpha z_j \otimes x^j &= \sum_{j=0}^{m-1} z_j \otimes x^j \alpha \\ &= \sum_{j=0}^{m-1} z_j \otimes \left(\sum_{i=0}^j (-1)^{j-i} \binom{j}{i} \rho^{-j} D^{j-i}(\alpha) x^i \right) \\ &= \sum_{j=0}^{m-1} z_j \left(\sum_{i=0}^j (-1)^{j-i} \binom{j}{i} \rho^{-j} D^{j-i}(\alpha) \right) \otimes x^i \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{m-1} \left(\sum_{j=i}^{m-1} (-1)^{j-i} \binom{j}{i} z_j \rho^{-j} D^{j-i}(\alpha) \right) \otimes x^i \\
&= \sum_{j=0}^{m-1} \left(\sum_{i=j}^{m-1} (-1)^{i-j} \binom{i}{j} z_i \rho^{-i} D^{i-j}(\alpha) \right) \otimes x^j.
\end{aligned}$$

Hence we obtain

$$\alpha z_j = \sum_{i=j}^{m-1} (-1)^{i-j} \binom{i}{j} z_i \rho^{-i} D^{i-j}(\alpha) \quad (0 \leq j \leq m-1).$$

Now we let $h = z_{m-1}$. Obviously, h is in V_{m-1} . Since $x \left(\sum_{j=0}^{m-1} z_j \otimes x^j \right) = \left(\sum_{j=0}^{m-1} z_j \otimes x^j \right) x$ and $x^m = -\sum_{j=0}^{m-1} x^j a_j = -\sum_{j=0}^{m-1} a_j x^j$, we see that

$$\begin{aligned}
\sum_{j=0}^{m-1} x z_j \otimes x^j &= \sum_{j=0}^{m-1} z_j \otimes x^{j+1} \\
&= \sum_{j=0}^{m-2} z_j \otimes x^{j+1} + h \otimes x^m \\
&= \sum_{j=1}^{m-1} z_{j-1} \otimes x^j + h \otimes \left(-\sum_{j=0}^{m-1} a_j x^j \right) \\
&= \sum_{j=1}^{m-1} z_{j-1} \otimes x^j - \sum_{j=0}^{m-1} h a_j \otimes x^j \\
&= -h a_0 \otimes 1 + \sum_{j=1}^{m-1} (z_{j-1} - h a_j) \otimes x^j.
\end{aligned}$$

It follows that $x z_j = z_{j-1} - h a_j$ ($1 \leq j \leq m-1$) and $x z_0 = -h a_0$. Noting that $h a_j = \rho^{m-1}(a_j) h = a_j h$ ($0 \leq j \leq m-1$), we have $z_{j-1} = x z_j + a_j h$ ($1 \leq j \leq m-1$) and $x z_0 = -a_0 h$. This implies $z_j = y_j h$ ($0 \leq j \leq m-1$), inductively.

Conversely, assume that h is in V_{m-1} . To show $\sum_{j=0}^{m-1} y_j h \otimes x^j \in (A \otimes_B A)^A$, it is suffice to prove that $x \left(\sum_{j=0}^{m-1} y_j h \otimes x^j \right) = \left(\sum_{j=0}^{m-1} y_j h \otimes x^j \right) x$ and $\alpha \left(\sum_{j=0}^{m-1} y_j h \otimes x^j \right) = \left(\sum_{j=0}^{m-1} y_j h \otimes x^j \right) \alpha$ for any $\alpha \in B$. Noting that $x y_j = y_{j-1} - a_j$ ($1 \leq j \leq m-1$) and $x y_0 = -a_0$, we obtain

$$\begin{aligned}
x \left(\sum_{j=0}^{m-1} y_j h \otimes x^j \right) &= x y_0 h \otimes 1 + \sum_{j=1}^{m-1} x y_j h \otimes x^j \\
&= (-a_0) h \otimes 1 + \sum_{j=1}^{m-1} (y_{j-1} - a_j) h \otimes x^j
\end{aligned}$$

$$\begin{aligned}
&= h \otimes (-a_0) + \sum_{j=1}^{m-1} (-a_j) h \otimes x^j + \sum_{j=1}^{m-1} y_{j-1} h \otimes x^j \\
&= h \otimes (-a_0) + h \otimes \left(-\sum_{j=1}^{m-1} x^j a_j \right) + \sum_{j=0}^{m-2} y_j h \otimes x^{j+1} \\
&= h \otimes x^m + \sum_{j=0}^{m-2} y_j h \otimes x^{j+1} \\
&= \left(\sum_{j=0}^{m-1} y_j h \otimes x^j \right) x.
\end{aligned}$$

Next, let α be an arbitrary element in B . Then we have

$$\begin{aligned}
\left(\sum_{j=0}^{m-1} y_j h \otimes x^j \right) \alpha &= \sum_{j=0}^{m-1} y_j h \otimes \left(\sum_{i=0}^j (-1)^{j-i} \binom{j}{i} \rho^{-j} D^{j-i}(\alpha) x^i \right) \\
&= \sum_{j=0}^{m-1} y_j h \left(\sum_{i=0}^j (-1)^{j-i} \binom{j}{i} \rho^{-j} D^{j-i}(\alpha) \right) \otimes x^i \\
&= \sum_{i=0}^{m-1} \left(\sum_{j=i}^{m-1} (-1)^{j-i} \binom{j}{i} y_j \rho^{m-j-1} D^{j-i}(\alpha) \right) h \otimes x^i \\
&= \sum_{j=0}^{m-1} \left(\sum_{i=j}^{m-1} (-1)^{i-j} \binom{i}{j} y_i \rho^{m-i-1} D^{i-j}(\alpha) \right) h \otimes x^j.
\end{aligned}$$

Hence to show $\alpha \left(\sum_{j=0}^{m-1} y_j h \otimes x^j \right) = \left(\sum_{j=0}^{m-1} y_j h \otimes x^j \right) \alpha$, it suffices to prove that

$$\alpha y_j = \sum_{i=j}^{m-1} (-1)^{i-j} \binom{i}{j} y_i \rho^{m-i-1} D^{i-j}(\alpha) \quad (0 \leq j \leq m-1).$$

We shall show it by induction. Since $y_{m-1} = 1$, it is true when $j = m-1$. For some j ($0 \leq j \leq m-2$), we assume that

$$\alpha y_{j+1} = \sum_{i=j+1}^{m-1} (-1)^{i-j-1} \binom{i}{j+1} y_i \rho^{m-i-1} D^{i-j-1}(\alpha).$$

Now we recall that $\alpha a_j = \sum_{i=j}^m (-1)^{i-j} \binom{i}{j} a_i \rho^{m-i} D^{i-j}(\alpha)$ ($0 \leq j \leq m$, $a_m = 1$) by Corollary 2.1. Noting that $xy_i = y_{i-1} - a_i$ ($1 \leq i \leq m-1$), we have

$$\begin{aligned}
\alpha y_j &= \alpha x y_{j+1} + \alpha a_{j+1} \\
&= x \rho(\alpha) y_{j+1} + D(\alpha) y_{j+1} + \alpha a_{j+1} \\
&= x \left(\sum_{i=j+1}^{m-1} (-1)^{i-j-1} \binom{i}{j+1} y_i \rho^{m-i-1} D^{i-j-1}(\rho(\alpha)) \right)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i=j+1}^{m-1} (-1)^{i-j-1} \binom{i}{j+1} y_i \rho^{m-i-1} D^{i-j-1}(D(\alpha)) \\
& + \sum_{i=j+1}^m (-1)^{i-j-1} \binom{i}{j+1} a_i \rho^{m-i} D^{i-j-1}(\alpha) \\
= & \sum_{i=j+1}^{m-1} (-1)^{i-j-1} \binom{i}{j+1} (y_{i-1} - a_i) \rho^{m-i} D^{i-j-1}(\alpha) \\
& - \sum_{i=j+1}^{m-1} (-1)^{i-j} \binom{i}{j+1} y_i \rho^{m-i-1} D^{i-j}(\alpha) \\
& + \sum_{i=j+1}^m (-1)^{i-j-1} \binom{i}{j+1} a_i \rho^{m-i} D^{i-j-1}(\alpha) \\
= & \sum_{i=j+1}^{m-1} (-1)^{i-j-1} \binom{i}{j+1} y_{i-1} \rho^{m-i} D^{i-j-1}(\alpha) \\
& - \sum_{i=j+1}^{m-1} (-1)^{i-j-1} \binom{i}{j+1} a_i \rho^{m-i} D^{i-j-1}(\alpha) \\
& - \sum_{i=j+1}^{m-1} (-1)^{i-j} \binom{i}{j+1} y_i \rho^{m-i-1} D^{i-j}(\alpha) \\
& + \sum_{i=j+1}^m (-1)^{i-j-1} \binom{i}{j+1} a_i \rho^{m-i} D^{i-j-1}(\alpha) \\
= & \sum_{i=j}^{m-2} (-1)^{i-j} \binom{i+1}{j+1} y_i \rho^{m-i-1} D^{i-j}(\alpha) \\
& - \sum_{i=j+1}^{m-1} (-1)^{i-j} \binom{i}{j+1} y_i \rho^{m-i-1} D^{i-j}(\alpha) \\
& + (-1)^{m-j-1} \binom{m}{j+1} D^{m-j-1}(\alpha) \\
= & y_j \rho^{m-j-1}(\alpha) \\
& + \sum_{i=j+1}^{m-2} (-1)^{i-j} \left\{ \binom{i+1}{j+1} - \binom{i}{j+1} \right\} y_i \rho^{m-i-1} D^{i-j}(\alpha) \\
& + (-1)^{m-j-1} \left\{ \binom{m}{j+1} - \binom{m-1}{j+1} \right\} D^{m-j-1}(\alpha) \\
= & y_j \rho^{m-j-1}(\alpha)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i=j+1}^{m-2} (-1)^{i-j} \binom{i}{j} y_i \rho^{m-i-1} D^{i-j}(\alpha) \\
& + (-1)^{m-j-1} \binom{m-1}{j} D^{m-j-1}(\alpha) \\
& = \sum_{i=j}^{m-1} (-1)^{i-j} \binom{i}{j} y_i \rho^{m-i-1} D^{i-j}(\alpha).
\end{aligned}$$

This completes the proof of Lemma 2.1. $\square$

Finally we shall prove Proposition 1.3 and Proposition 1.4.

Proof of Proposition 1.3. It is obvious by Lemma 1.1 and Lemma 2.1. $\square$

Proof of Proposition 1.4. Let f be Hirata separable in $B[X; \rho, D]$. It follows from Lemma 1.2 and Lemma 2.1 that there exist $g_i \in V_0$ and $\sum_{j=0}^{m-1} y_j h_i \otimes x^j \in (A \otimes_B A)^A$ with $h_i \in V_{m-1}$ such that

$$1 \otimes 1 = \sum_i g_i \sum_{j=0}^{m-1} y_j h_i \otimes x^j = \sum_{j=0}^{m-1} \left(\sum_i g_i y_j h_i \right) \otimes x^j.$$

This implies

$$\sum_i g_i y_0 h_i = 1 \quad \text{and} \quad \sum_i g_i y_k h_i = 0 \quad (1 \leq k \leq m-1).$$

Then we obtain inductively,

$$\sum_i g_i x^k h_i = 0 \quad (0 \leq k \leq m-2) \quad \text{and} \quad \sum_i g_i x^{m-1} h_i = 1.$$

Conversely, let g_i be in V_0 and h_i in V_{m-1} such that $\sum_i g_i x^{m-1} h_i = 1$ and $\sum_i g_i x^k h_i = 0$ ($0 \leq k \leq m-2$). Note that $\sum_{j=0}^{m-1} y_j h_i \otimes x^j \in (A \otimes_B A)^A$ by Lemma 2.1. By an easy induction, we see that

$$\sum_i g_i y_k h_i = 0 \quad (1 \leq k \leq m-1) \quad \text{and} \quad \sum_i g_i y_0 h_i = 1.$$

This implies

$$\sum_i g_i \sum_{j=0}^{m-1} y_j h_i \otimes x^j = \sum_{j=0}^{m-1} \left(\sum_i g_i y_j h_i \right) \otimes x^j = 1 \otimes 1.$$

Therefore, f is Hirata separable in $B[X; \rho, D]$ by Lemma 1.2. This completes the proof of Proposition 1.4. $\square$

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