

RICCI SOLITONS ON TANGENT BUNDLES WITH THE COMPLETE LIFT OF A PROJECTIVE SEMI-SYMMETRIC CONNECTION

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ABSTRACT. In this paper, we investigate the Ricci solitons on tangent bundles with respect to the complete lift of a projective semi-symmetric connection using vertical and complete lifts of torqued vector fields.

1. INTRODUCTION AND PRELIMINARIES

1.1. Ricci solitons, torse forming vector fields and projective semi-symmetric connections. Let (M, g) be an n -dimensional (pseudo-)Riemannian manifold. A Ricci soliton (M, g, τ, λ) is defined by the condition

$$Ric + \frac{1}{2}\mathfrak{L}_\tau g = \lambda g, \quad (1.1)$$

where Ric is the Ricci tensor of the metric g , $\mathfrak{L}_\tau$ is the Lie differentiation with respect to the potential field τ on M and λ is a real constant. A Ricci soliton is said to be shrinking if $\lambda > 0$, expanding if $\lambda < 0$ and steady if $\lambda = 0$. Ricci solitons generalize the Einstein metrics.

A nonvanishing vector field τ on a (pseudo-)Riemannian manifold (M, g) is said to be torse-forming if

$$\nabla_U \tau = \phi U + \alpha(U)\tau, \quad (1.2)$$

where U is a vector field, ϕ is a smooth function, ∇ is the Levi-Civita connection of g and α is a 1-form, [15]. Furthermore, we call the vector field τ as concircular ([4],[13]) if the 1-form $\alpha = 0$ in equation (1.2). Moreover, if the function $\phi = 1$, we say that the vector field τ is concurrent, ([10],[14]). If the function $\phi = 0$ in (1.2), we say the vector field τ is recurrent. If $\phi = \alpha = 0$ in (1.2), then we call the vector field τ as parallel. Chen [3] defined a new vector field, which is called a torqued vector field τ , as satisfying (1.2) with $\alpha(\tau) = 0$. In this case, the 1-form α denotes the torqued form of τ and ϕ denotes the torqued function.

From [12], a semi-symmetric metric connection $\tilde{\nabla}$ on M is given by

$$\tilde{\nabla}_U V = \nabla_U V + \pi(V)U - g(U, V)P, \quad (1.3)$$

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where $\pi(U) = g(U, P)$, $\forall U \in \chi(M)$, the Lie algebra of vector fields of M .

From [17], a projective semi-symmetric connection $\bar{\nabla}$ on M is given by

$$\bar{\nabla}_U V = \nabla_U V + \psi(V)U + \psi(U)V + \mu(V)U - \mu(U)V, \quad (1.4)$$

where ψ and μ are 1-forms satisfying $\psi(U) = \frac{n-1}{2(n+1)}\pi(U)$, $\mu(U) = \frac{1}{2}\pi(U)$.

For the Riemannian curvature tensor $\bar{R}$, Ricci tensor $\bar{Ric}$ and the scalar curvature $\bar{r}$ of $\bar{\nabla}$, we have

$$\begin{aligned} \bar{R}(U, V)W &= R(U, V)W + \theta(U, V)W + \omega(U, W)V - \omega(V, W)U, \\ \bar{Ric}(V, W) &= Ric(V, W) + \theta(V, W) - (n-1)\omega(V, W), \\ \bar{r} &= r + Tr(\theta) - (n-1)Tr(\omega), \end{aligned} \quad (1.5)$$

where $\theta(U, V) = \frac{1}{2}[(\nabla_V \pi)(U) - (\nabla_U \pi)(V)]$, $\omega(U, V) = \frac{n-1}{2(n+1)}(\nabla_U \pi)(V) + \frac{1}{2}(\nabla_V \pi)(U) - \frac{n^2}{(n+1)^2}\pi(U)\pi(V)$, for all $U, V, W \in \chi(M)$. (see also [9]).

1.2. Tangent bundle. In this subsection, some basic facts will be repeated about tangent bundles of differentiable manifolds that be used in the future (for more information see [11]). Given an n -dimensional Riemannian manifold (M, g) . The tangent bundle TM of the manifold M is $TM = \cup_{p \in M} T_p M$, where $T_p M$ is the tangent space of M at p , and the manifold TM is $2n$ -dimensional which has a differentiable structure induced from M .

Let f, U, ω be a function, a vector field, a 1-form on M , respectively. We will denote their vertical lifts by f^v, U^v, ω^v and their complete lifts by f^c, U^c, ω^c to tangent bundle TM . The following relations are satisfied:

$$\begin{aligned} (fU)^v &= f^v U^v, \quad U^v f^v = 0, \quad \omega^v(U^v) = 0, \\ (fU)^c &= f^c U^v + f^v U^c, \quad U^c f^c = (Uf)^c, \quad \omega^c(U^c) = (\omega(U))^c, \\ U^v(f^c) &= U^c(f^v) = (Uf)^v, \quad \omega^v(U^c) = \omega^c(U^v) = (\omega(U))^v. \end{aligned} \quad (1.6)$$

For the Lie brackets, we have

$$[U^v, V^v] = 0, \quad [U^v, V^c] = [U, V]^v, \quad [U^c, V^c] = [U, V]^c.$$

We know that that the complete lift metric g^c of the metric g is a semi-Riemannian metric on TM and it is given by

$$\begin{cases} g^c(U^v, V^c) = g^c(U^c, V^v) = (g(U, V))^v, \\ g^c(U^v, V^v) = 0, \\ g^c(U^c, V^c) = (g(U, V))^c, \forall U, V \in \chi(M). \end{cases} \quad (1.7)$$

The complete lift connection ∇^c of a linear connection ∇ fulfills the following relations:

$$\nabla_{U^v}^c V^v = (\nabla_U V)^v, \quad \nabla_{U^c}^c V^c = (\nabla_U V)^c, \quad \nabla_{U^v}^c V^c = \nabla_{U^c}^c V^v = (\nabla_U V)^v. \quad (1.8)$$

For some significant recent papers about lifting theory on tangent bundles, we may refer to ([1, 2, 6, 7, 8, 16]).

According to Proposition 3.11 in [11] and using (1.5), we have the following proposition.

Proposition 1.1. *Given a Riemannian manifold (M, g) and its tangent bundle TM with the complete lift metric g^c . Then the relation between the Ricci tensor Ric of g and the Ricci tensor Ric^c of the metric g^c is given by*

$$Ric^c(U^v, V^v) = 0, \quad Ric^c(U^v, V^c) = (Ric(U, V))^v, \quad Ric^c(U^c, V^c) = (Ric(U, V))^c \quad (1.9)$$

for all $U, V \in \chi(M)$.

From (1.5) and Proposition 1.1, we have

Proposition 1.2. *Given an n -dimensional Riemannian manifold (M, g) endowed with the projective semi-symmetric connection $\bar{\nabla}$ defined by (1.4), and $(TM, g^c, \bar{\nabla}^c)$ its tangent bundle with the complete lift metric g^c and the complete lift of the projective semi-symmetric connection $\bar{\nabla}^c$. In this case, the Ricci tensor field $\bar{Ric}^c$ of $\bar{\nabla}^c$ fulfills*

$$\begin{aligned} \bar{Ric}^c(U^v, V^c) &= (\bar{Ric}(U, V))^v = (Ric(U, V))^v + (\theta(U, V))^v - (n-1)(\omega(U, V))^v, \\ \bar{Ric}^c(U^c, V^c) &= (\bar{Ric}(U, V))^c = (Ric(U, V))^c + (\theta(U, V))^c - (n-1)(\omega(U, V))^c. \end{aligned} \quad (1.10)$$

In this paper, we study Ricci solitons on tangent bundles using vertical and complete lifts of torqued vector fields and considering the complete lift of a projective semi-symmetric connection. Similar problems were examined in [5] with respect to a semi-symmetric metric connection.

2. RICCI SOLITONS ON TANGENT BUNDLES

In this main section, we examine the Ricci solitons on the tangent bundles of Riemannian manifolds when the potential is the vertical lift and the complete lift of a torqued vector field, respectively. The tangent bundle will be endowed with the complete lift of a projective semi-symmetric connection.

2.1. Case 1: The potential is vertical lift of a vector field.

Proposition 2.1. *Let (M, g, ∇, τ) be a Riemannian manifold with a torqued vector field τ . Then, $(TM, g^c, \bar{\nabla}^c, \tau^v, \lambda)$ is a Ricci soliton if and only if the following conditions hold:*

- i) $Ric(U, V) = \bar{\lambda}g(U, V) - \theta(U, V) + (n-1)\omega(U, V)$,
 - ii) $\frac{1}{n(n+1)}\pi(\tau)g - \frac{1}{2}[\frac{1}{n+1}(\pi - \alpha) \otimes \eta + \eta \otimes \frac{1}{n+1}(\pi - \alpha)] = 0$,
 - iii) The function ϕ in (1.2) is $\frac{1-m}{m}\pi(\tau)$.
- Here η denotes the dual 1-form of τ .

Proof. First we compute $(\mathfrak{L}_{\tau^v}g^c)(\tilde{U}, \tilde{V})$ and $(\bar{\mathfrak{L}}_{\tau^v}g^c)(\tilde{U}, \tilde{V})$ for all vector fields $\tilde{U}, \tilde{V}$ on TM . From (1.2) and (1.8), we get

$$\begin{aligned} (\mathfrak{L}_{\tau^v}g^c)(U^c, V^c) &= [g(\nabla_U \tau, V) + g(\nabla_V \tau, U)]^v \\ &= [2\phi g(U, V) + \alpha(U)g(V, \tau) + \alpha(V)g(U, \tau)]^v, \\ (L_{\tau^v}g^c)(U^c, V^v) &= 0. \end{aligned} \quad (2.1)$$

From (1.2), (1.4), (1.8) and (2.1), we obtain

$$\begin{aligned}
 (\bar{\mathfrak{L}}_{\tau^v} g^c)(U^c, V^c) &= [g(\bar{\nabla}_U \tau, V) + g(\bar{\nabla}_V \tau, U)]^v \\
 &= [2(\phi + \frac{n}{n+1}\pi(\tau))g(U, V) + (\alpha(U) - \frac{1}{n+1}\pi(U))g(V, \tau) \\
 &\quad + (\alpha(V) - \frac{1}{n+1}\pi(V))g(U, \tau)]^v,
 \end{aligned} \tag{2.2}$$

$$(\bar{\mathfrak{L}}_{\tau^v} g^c)(U^c, V^v) = 0.$$

Now, suppose that $(TM, g^c, \bar{\nabla}^c, \tau^v, \bar{\lambda})$ is a Ricci soliton. Then, we have

$$\bar{Ric}^c(U^c, V^c) = \bar{\lambda}g^c(U^c, V^c) - \frac{1}{2}(\bar{\mathfrak{L}}_{\tau^v} g^c)(U^c, V^c).$$

Using (1.10-2.2), we get

$$\begin{aligned}
 (Ric(U, V) + \theta(U, V) - (n-1)\omega(U, V))^c &= \bar{\lambda}(g(U, V))^c \\
 &\quad - \frac{1}{2}[2(\phi + \frac{n}{n+1}\pi(\tau))g(U, V) \\
 &\quad + (\alpha(U) - \frac{1}{n+1}\pi(U))g(V, \tau) \\
 &\quad + (\alpha(V) - \frac{1}{n+1}\pi(V))g(U, \tau)]^v.
 \end{aligned}$$

Thus, we occur

$$Ric(U, V) = \bar{\lambda}g(U, V) - \theta(U, V) + (n-1)\omega(U, V),$$

$$(\phi + \frac{n}{n+1}\pi(\tau))g(U, V) = \frac{1}{2}[(\frac{1}{n+1}\pi - \alpha)(U)g(V, \tau) + (\frac{1}{n+1}\pi - \alpha)(V)g(U, \tau)]. \tag{2.3}$$

If we take contraction over U and V in (2.3), we obtain

$$(\phi + \frac{n}{n+1}\pi(\tau))n = \frac{1}{n+1}(\pi(\tau) - \alpha(\tau)).$$

Since for a torqued vector field $\alpha(\tau) = 0$, above equation gives

$$\phi = \frac{1-n}{n}\pi(\tau).$$

If we denote the dual 1-form of τ by η , we occur from (2.3)

$$\frac{1}{n(n+1)}\pi(\tau)g - \frac{1}{2}[\frac{1}{n+1}(\pi - \alpha) \otimes \eta + \eta \otimes \frac{1}{n+1}(\pi - \alpha)] = 0.$$

Contrarily, let (M, g, ∇, τ) be a Riemannian manifold with a torqued vector field τ . From (1.5), (1.9), (1.10) and (2.2), we have

$$\begin{aligned} \overline{Ric}^c(U^c, V^c) + \frac{1}{2}(\overline{\mathfrak{L}}_{\tau^v}g^c)(U^c, V^c) &= (\overline{Ric}(U, V))^c + (\phi g(U, V))^v + \left(\frac{n}{n+1}\pi(\tau)g(U, V)\right)^v \\ &\quad + \frac{1}{2}\left[\left(\alpha - \frac{1}{n+1}\pi\right)(U)g(V, \tau) \right. \\ &\quad \left. + \left(\alpha - \frac{1}{n+1}\pi\right)(V)g(U, \tau)\right]^v \\ &= [Ric(U, V) + \theta(U, V) - (n-1)\omega(U, V)]^c \\ &\quad + \left[\phi + \frac{n-1}{n}\pi(\tau)\right]^v(g(U, V))^v, \end{aligned}$$

$$\begin{aligned} \overline{Ric}^c(U^c, V^v) + \frac{1}{2}(\overline{\mathfrak{L}}_{\tau^v}g^c)(U^c, V^v) &= (\overline{Ric}(U, V))^v \\ &= [Ric(U, V) + \theta(U, V) - (n-1)\omega(U, V)]^v. \end{aligned}$$

It is clear that if $Ric(U, V) = \bar{\lambda}g(U, V) - \theta(U, V) + (n-1)\omega(U, V)$ and $\phi = \frac{1-n}{n}\pi(\tau)$, then $(TM, g^c, \bar{\nabla}^c, \tau^v, \bar{\lambda})$ is a Ricci soliton. This completes the proof. $\square$

2.2. Case 2: The potential is complete lift of a vector field. In the following propositions, we give some characterizations about Ricci solitons on TM by replacing the vector field τ^h with τ^c in the above proposition.

Proposition 2.2. *Let (M, g, ∇, τ) be a Riemannian manifold with a torqued vector field τ . Then, $(TM, g^c, \bar{\nabla}^c, \tau^c, \bar{\lambda})$ is a Ricci soliton if the following condition holds:*

$$\begin{aligned} Ric(U, V) &= (\bar{\lambda} - \phi - \frac{1}{n+1}\pi(\tau))g(U, V) \\ &\quad + \frac{1}{2}\left[\left(\frac{1}{n+1}\pi - \alpha\right)(U)g(V, \tau) + \left(\frac{1}{n+1}\pi - \alpha\right)(V)g(U, \tau)\right] \\ &\quad - \theta(U, V) + (n-1)\omega(U, V), \end{aligned}$$

where Ric denotes the Ricci tensor of ∇ .

Proof. First we compute $(\overline{\mathfrak{L}}_{\tau^c}g^c)(\tilde{U}, \tilde{V})$ for all vector fields $\tilde{U}, \tilde{V}$ on TM :

$$\begin{aligned} (\overline{\mathfrak{L}}_{\tau^c}g^c)(U^c, V^c) &= [g(\bar{\nabla}_U\tau, V) + g(\bar{\nabla}_V\tau, U)]^c \tag{2.4} \\ &= \left[2\left(\phi + \frac{n}{n+1}\pi(\tau)\right)g(U, V) + \left(\alpha - \frac{1}{n+1}\pi\right)(U)g(V, \tau) \right. \\ &\quad \left. + \left(\alpha - \frac{1}{n+1}\pi\right)(V)g(U, \tau)\right]^c, \\ (\overline{\mathfrak{L}}_{\tau^c}g^c)(U^c, V^v) &= [g(\bar{\nabla}_U\tau, V) + g(\bar{\nabla}_V\tau, U)]^v \\ &= \left[2\left(\phi + \frac{n}{n+1}\pi(\tau)\right)g(U, V) + \left(\alpha - \frac{1}{n+1}\pi\right)(U)g(V, \tau) \right. \\ &\quad \left. + \left(\alpha - \frac{1}{n+1}\pi\right)(V)g(U, \tau)\right]^v, \end{aligned}$$

Using (1.1) and (2.4), we get

$$\begin{aligned}
\overline{Ric}^c(U^c, V^c) &= \bar{\lambda}g^c(U^c, V^c) - \frac{1}{2}(\bar{\mathfrak{L}}_{\tau^c}g^c)(U^c, V^c). \\
(\overline{Ric}(U, V))^c &= \bar{\lambda}(g(U, V))^c - \frac{1}{2}[2(\phi + \frac{n}{n+1}\pi(\tau))g(U, V)]^c \\
&\quad - \frac{1}{2}[(\alpha - \frac{1}{n+1}\pi)(U)g(V, \tau) + (\alpha - \frac{1}{n+1}\pi)(V)g(U, \tau)]^c \\
&= [(\bar{\lambda} - \phi - \frac{1}{n+1}\pi(\tau))g(U, V)]^c \\
&\quad + \frac{1}{2}[(\frac{1}{n+1}\pi - \alpha)(U)g(V, \tau) + (\frac{1}{n+1}\pi - \alpha)(V)g(U, \tau)]^c.
\end{aligned}$$

From (1.5), we have

$$\begin{aligned}
(Ric(U, V) + \theta(U, V) - (n-1)\omega(U, V))^c &= [(\bar{\lambda} - \phi - \frac{1}{n+1}\pi(\tau))g(U, V)]^c \\
&\quad + \frac{1}{2}[(\frac{1}{n+1}\pi - \alpha)(U)g(V, \tau) \\
&\quad + (\frac{1}{n+1}\pi - \alpha)(V)g(U, \tau)]^c
\end{aligned}$$

which implies

$$\begin{aligned}
Ric(U, V) &= (\bar{\lambda} - \phi - \frac{1}{n+1}\pi(\tau))g(U, V) + \frac{1}{2}((\frac{1}{n+1}\pi - \alpha)(U)g(V, \tau) + \\
&\quad (\frac{1}{n+1}\pi - \alpha)(V)g(U, \tau)) - \theta(U, V) + (n-1)\omega(U, V).
\end{aligned}$$

□

Proposition 2.3. *Let (M, g, ∇, τ) be a Ricci soliton with a torqued vector field τ . Then $(TM, g^c, \bar{\nabla}^c, \tau^c, \bar{\lambda})$ is a Ricci soliton if the following conditions hold:*

i) ϕ in (1.2) is a constant.

ii) $\theta(U, V) - (n-1)\omega(U, V) = -\frac{1}{2}(\alpha - \pi)(U)g(V, \tau) + \frac{1}{2}(\alpha - \pi)(V)g(U, \tau)$ for all $U, V \in \chi(M)$.

Proof. Let (M, g, ∇, τ) be a Ricci soliton with a torqued vector field τ . Then, we have

$$(\mathfrak{L}_\tau g)(U, V) = 2\phi g(U, V) + \alpha(U)g(\tau, V) + \alpha(V)g(\tau, U) \quad (2.5)$$

for all $U, V \in \chi(M)$. Using (1.1) and (2.5), we obtain

$$Ric(U, V) = (\lambda - \alpha)g(U, V) - \frac{1}{2}\alpha(U)g(\tau, V) - \frac{1}{2}\alpha(V)g(\tau, U).$$

On the other hand, we compute

$$\begin{aligned}
(\bar{\mathfrak{L}}_\tau g)(U, V) &= g(\bar{\nabla}_U \tau, V) + g(\bar{\nabla}_V \tau, U) \\
&= (\mathfrak{L}_\tau g)(U, V) + \frac{1}{n+1}[2n\pi g(U, V) - (g(U, \tau)\pi(V) + g(V, \tau)\pi(U))] \\
&= 2(\phi + \frac{n}{n+1})g(U, V) + (\alpha - \pi)(U)g(V, \tau) + (\alpha - \pi)(V)g(U, \tau).
\end{aligned} \quad (2.6)$$

Equations (1.5), (1.9) and (2.6) give us

$$\begin{aligned}
\overline{Ric}^c(U^c, V^c) + \frac{1}{2}(\overline{\mathfrak{L}}_{\tau^c}g^c)(U^c, V^c) &= (\overline{Ric}(U, V))^c + \frac{1}{2}((\overline{\mathfrak{L}}_{\tau}g)(U, V))^c \\
&= (Ric(U, V) + \theta(U, V) - (n-1)\omega(U, V))^c \\
&\quad + ((\phi + \frac{n}{n+1})g(U, V) + \frac{1}{2}(\alpha - \pi)(U)g(V, \tau) \\
&\quad + \frac{1}{2}(\alpha - \pi)(V)g(U, \tau))^c \\
&= (\lambda g(U, V) + \theta(U, V) - (n-1)\omega(U, V))^c \\
&\quad + ((\phi + \frac{n}{n+1})g(U, V) + \frac{1}{2}(\alpha - \pi)(U)g(V, \tau) \\
&\quad + \frac{1}{2}(\alpha - \pi)(V)g(U, \tau))^c \\
&= (\lambda + \phi + \frac{n}{n+1})(g(U, V))^c + (\theta(U, V) \\
&\quad - (n-1)\omega(U, V))^c + \frac{1}{2}((\alpha - \pi)(U)g(V, \tau) \\
&\quad + \frac{1}{2}(\alpha - \pi)(V)g(U, \tau))^c.
\end{aligned}$$

So, in order that $\overline{Ric}^c(U^c, V^c) + \frac{1}{2}(\overline{\mathfrak{L}}_{\tau^c}g^c)(U^c, V^c) = \bar{\lambda}g(U, V)$, we find $\bar{\lambda} = \lambda + \phi + \frac{n}{n+1}$, ϕ is a constant and $\theta(U, V) - (n-1)\omega(U, V) = -\frac{1}{2}(\alpha - \pi)(U)g(V, \tau) + \frac{1}{2}(\alpha - \pi)(V)g(U, \tau)$. $\square$

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