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AMALGAMATED RINGS WITH SEMI NIL-CLEAN PROPERTIES

VIJAYANAND VENKATACHALAM[1] AND SELVARAJ CHELLIAH[2*]

ABSTRACT. An element in a ring R is said to be semi nil-clean if it is a sum of nilpotent and periodic elements in R. An element in a ring R is said to be semiclean if it is a sum of unit and periodic elements in R. A ring R is said to be semi nil-clean (resp., semiclean) if every element in R is semi nil-clean (resp., semiclean). We discuss some basic properties of semi nil-clean ring and we also study the concepts of semi nil-clean and semiclean properties in various context of commutative rings such as amalgamations and the ring of polynomials $R[x]$ of a ring R.

1. INTRODUCTION AND PRELIMINARIES

The purpose of this article is to study rings, usually commutative, in which every element is the sum of a nilpotent n (i.e.,$n^k = 0$, for some k) and a periodic element p (i.e., $p^m = p^l$ for some $1 \leq l < m$). We call an element semi nil-clean if it is the sum of a nilpotent and a periodic element and call a ring semi nil-clean if each element is semi nil-clean. Nicholson [14], defined a ring (not necessarily commutative) to be clean if every element is the sum of a unit and and an idempotent. In [16], Ye defined a ring to be semiclean if every element is the sum of a unit and periodic element, as a generalization of the notion clean ring. In [9], Diesl introduced a new class of rings called nil-clean rings, in which every element is a sum of nilpotent and idempotent elements. In this paper, we give generalization to this ring called semi nil-clean. Every nil-clean element is semi nil-clean element. Every unit element in a semi nil-clean ring is unipotent. The following diagram of the implications summarizes the relations between the main notions involved in this paper.

$$\begin{tikzcd}
 & & \boxed{+\text{UU}} \arrow[dd, Rightarrow] \arrow[dr, Rightarrow, no head] & \\
 & & & \boxed{\text{Semiclean}} \\
\boxed{\text{Nil clean}} \arrow[r, Rightarrow] & \boxed{\text{Periodic}} \arrow[r, Rightarrow] & \boxed{\text{Semi nil-clean}} \arrow[ur, Rightarrow] &
\end{tikzcd}$$

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* Corresponding author.
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For the convenience of the reader, we denote by $U(R)$, $Idem(R)$, $Nilp(R)$, $Per(R)$ and $Jac(R)$, the set of all unit elements of ring R, the set of all idempotent elements of ring R, the ideal of all nilpotent elements of ring R, the set of all periodic elements of ring R and the Jacobson radical of R, respectively. Recall that a ring R is UU-ring if every unit $u \in R$ is unipotent, that is, u can be expressed as $u = 1 + n$, where n is a nilpotent element of ring R. The trivial ring extension of R by M be an $R-$module. Let R and S be commutative rings, J be an ideal of S and $f : R \to S$ be a ring homomorphism. In this setting, we can consider the following subring of $R \times S$:

$$R \bowtie^f J := \{(r, f(r) + j) \mid r \in R, j \in J\}$$

called *the amalgamation of R with S along J with respect to f* (introduced and studied by D'Anna, Finocchiaro, and Fontana in ([2],[3]]). This construction is a generalization of *the amalgamated duplication of a ring along an ideal* (introduced and studied by D'Anna and Fontana in ([4], [5], [6])). Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation (see[2, Examples 2.5 and 2.6]). One of the key tools for studying $R \bowtie^f J$ is based on the fact that the amalgamation can be studied in the frame of pullback constructions [2, Section 4]. This point of view allows the authors in ([2],[3]) to provide an ample description of various properties of $R \bowtie^f J$, in connection with the properties of R, J and f. In [2], the authors studied the basic properties of this construction and they characterized those distinguished pullbacks that can be expresses as an amalgamation. Moreover, in [3] they pursued the investigation on the structure of the rings of the form $R \bowtie^f J$, with particular attention to the prime spectrum, to the claim properties and the Krull dimension. In [1], the authors studied the SIT-ring and SITT-ring properties in amalgamated and bi-amalgamated algebras along ideals.

Further, we also study the concepts of semi nil-clean and semiclean properties in various context of commutative rings such as amalgamations and the ring of polynomial $R[x]$ of a ring R

2. Semi nil-clean ring

Definition 2.1. Let R be a ring. An element $r \in R$ is called semi nil-clean if there exist a nilpotent $n \in R$ and a periodic element $p \in R$ such that $r = n + p$. A ring is called semi nil-clean if every element of the ring is semi nil-clean.

Before going further, we give some examples of semi nil-clean. It is clear that every idempotent, nilpotent and periodic elements are semi nil-clean elements. Therefore it is clear Boolean and periodic rings are semi nil-clean.

Proposition 2.2. *Every nil-clean ring is a semi nil-clean.*

The following example illustrate the converse of the Proposition 2.2 is not true.

Example 2.3. Let $\mathbb{Z}/3\mathbb{Z}$ be the ring of integers modulo 3. Then R is semi nil-clean, but not a nil-clean.

In the next proposition, we discuss the properties that a semi nil-clean element possess.

Proposition 2.4. *Let R be a commutative ring. Then*

(1) *If $r \in R$ is a semi nil-clean element, then the homomorphic image of r is also a semi nil-clean element.*
(2) *If $r \in R$ is a semi nil-clean element, then r^n is a nil-clean element.*
(3) *If $r \in R$ is a semi nil-clean element, then $r^n - r^{2n}$ is a nilpotent for some n.*

Proof. (1) This is immediate since the homorphic image of a nilpotent (resp., periodic) element is nilpotent (resp., periodic).
(2) Assume that $r \in R$ is a semi nil-clean element. Then, we have $r = n + p$ with $n \in Nilp(R)$ and $p \in Per(R)$. By [16, Lemma 5.2], there exists $k \in \mathbb{N}$ such that p^k is idempotent. This implies $(r-n)^k = p^k$ is idempotent. By applying Binomial theorem we get our desired result.
(3) Suppose $r \in R$ is a semi nil-clean element. Then by (2), r^n is a nil-clean element. Then by [12, Theorem 3], $r^n - r^{2n}$ is nilpotent.

□

Theorem 2.5. *Let $\{R_i : 1 \leq i \leq k\}$ be rings. Then the direct product $R = R_1 \times R_2 \times \cdots \times R_k$ is a semi nil-clean ring if and only if each R_i is semi nil-clean.*

Proof. ($\Rightarrow$) If the direct product R is semi nil-clean, then it is easy to see each R_i is semi nil-clean by Proposition 2.4(1).
($\Leftarrow$) Assume that each R_i is semi nil-clean. Let $r = (r_i) \in R$. Then for each i, $r_i = n_i + p_i$, where $n_i \in Nilp(R_i)$ and $p_i \in Per(R_i)$. Then $r = n + p$, where $n = (n_i)$ is nilpotent in R and $p = (p_i)$ is a periodic element in R by [15, Lemma 2.4] . Hence R is semi nil-clean.

□

The following theorem examines the semi nil-clean property to indecomposable rings. Recall that a ring R is indecomposable if 0 and 1 are the only idempotents of the ring R.

Theorem 2.6. *Let R be an indecomposable ring. Then every semi nil-clean element is either nilpotent or unit.*

Proof. Assume that R is an indecomposable ring and $r \in R$ is a semi nil-clean element. Then we have $r = n + p$, where $n \in Nilp(R)$ and $p \in Per(R)$. By[16, Lemma 5.2], p^k is idempotent for some $k > 1$. Since R is indecomposable, p^k is either 0 or 1. Suppose $p^k = 0$, then $(r-n)^k = 0$ implies that r^k is nilpotent. Otherwise, $p^k = 1$ implies $(r-n)^k = 1$. Therefore, r is either a nilpotent or a unit element in R.

□

Corollary 2.7. *Let A be an integral domain. Then every semi nil-clean element is either a unit or a nilpotent.*

Proof. This follows form Theorem 2.6.

□

Definition 2.8. [16] Let I be an ideal of a ring R. We say that periodics in R can be lifted modulo I if for any $r \in R$ with $r^k - r^l \in I, (k > l)$, there exists $p \in R$ such that $p^k = p^l$ and $r - p \in I$

Proposition 2.9. *Let R be a UU-ring and I be an ideal of R such that $I \subset Jac(R)$. If R/I is semi nil-clean and if periodics in R can be lifted modulo I, then R is semi nil-clean.*

Proof. let us use $\overline{r}$ to denote $r+I$ in R/I. Let $r \in R$, then we can write $\overline{r} = \overline{n}+\overline{p}$, where $\overline{n}$ is nilpotent and $\overline{p}^k = \overline{p}^l, (k > l)$ in R/I. By assumption, we may assume $p^k = p^l$. By the fact $\overline{1 + r - p} = \overline{1 + n}$ is a unit in R/I and $I \subseteq J(R)$. Now, it is easy to see $1 + r - p$ is a unit in R. This implies $r = n + p$.

□

Proposition 2.10. *Let R be a ring. Then the polynomial ring $R[x]$ is not a semi nil-clean ring.*

Proof. Assume that $x = n + p$, where $p^m = p^l, (m > l)$ and $p \in R[x]$ and n is nilpotent in $R[x]$. By the commutativity of R, p must be in R. Thus $-p + x$ is nilpotent in $R[x]$. This implies 1 is nilpotent. This is absurd.

□

The following propositions give some characterizations of semi nil-clean rings.

Proposition 2.11. *R is semi nil-clean ring if and only if For each $r \in R$, we have $r = u + p$ where $p \in Per(R)$ and $u \in UU(R)$*

Proof. ($\Rightarrow$) Assume that R is a semi nil-clean ring. Let $r \in R$. Then we have $r - 1 = n + p$, where $n \in Nilp(R)$ and $p \in Per(R)$. This implies $r = u + p$ such that $u = 1 + n \in UU(R)$ and $p \in Per(R)$.
($\Leftarrow$) Let us assume that each element $r \in R$, we have $r = u+p$, where $u \in UU(R)$ and $p \in Per(R)$. Let $r \in R$, then $r+1 = u+p$ with $u \in UU(R)$ and $p \in Per(R)$. This implies that $r + 1 = 1 + n + p$ such that $n \in Nil(R)$. Therefore, $r = n + p$. Hence, R is semi nil-clean.

□

Proposition 2.12. *Every semi nil-clean ring is a semiclean ring.*

Proof. Assume that R is semi nil-clean. Let $r \in R$, then $r - 1 = n + p$, where $n \in Nilp(R)$ and $p \in Per(R)$. This implies $r = 1 + n + p$. Therefore, $r = u + p$, with $1 + n = u \in U(R)$. Hence, R is semiclean.

□

The following example illustrate the converse of the proposition 2.12 is not true.

Example 2.13. Every infinite field is semiclean ring, but not a semi nil-clean.

Proposition 2.14. *If R is semiclean and UU-ring, then R is semi nil-clean ring.*

Proof. Assume that R is both semiclean and UU-ring. Let $r \in R$, then $r - 1 = u + p$ where $u \in U(R)$ and $p \in Per(R)$. Since R is UU-ring, $u = 1 + n$ for some $n \in Nilp(R)$. Therefore, $r = n + p$. Hence, R is semi nil-clean.

□

Proposition 2.15. *Let $f : R \to S$ be a ring homomorphism and J be an ideal of S. If $R \bowtie^f J$ is a semi nil-clean ring, then R and $f(R) + J$ are semi nil-clean rings.*

Proof. The rings R and $f(R) + J$ are proper homomorphic image of $R \bowtie^f J$. Then, by Proposition 2.4(1), they are semi nil-clean rings.

$\square$

Theorem 2.16. *Let $f : R \to S$ be a ring homomorphism and J be an ideal of S such that $J \subset Nilp(S)$. Then $R \bowtie^f J$ is a semi nil-clean ring if and only if R is semi nil-clean.*

Proof. ($\Rightarrow$) It follows from Proposition 2.42.4(1).
($\Leftarrow$) Assume that R is a semi nil-clean ring. Let $(r, f(r) + j) \in R \bowtie^f J$. Since R is a semi nil-clean we can write $r = n + p$, where $n \in Nilp(R)$ and $p \in Per(R)$. Then $(r, f(r)+j) = (n, f(n)+j)+(p, f(p))$, it is clear that $(p, f(p))$ is a periodic element in $R \bowtie^f J$. Then by [8, Lemma 2.10], $(n, f(n) + j) \in Nilp(R \bowtie^f J)$. Hence $R \bowtie^f J$ is a semi nil-clean ring.

$\square$

Theorem 2.17. *Let $f : R \to S$ be a ring homomorphism and J be an ideal of S. If S is periodic ring, then $R \bowtie^f J$ is a semi nil-clean ring if and only if R is semi nil-clean.*

Proof. ($\Rightarrow$) By Proposition 2.4(1), it is clear.
($\Leftarrow$) Assume that R is a semi nil-clean ring. Let $(r, f(r)+j) \in R \bowtie^f J$. Since R is a semi nil-clean we can write $r = n+p$, where $n \in Nilp(R)$ and $p \in Per(R)$. Then $(r, f(r)+j) = (n, f(n))+(p, f(p)+j)$. It is clear $(n, f(n)) \in Nilp(R \bowtie^f J)$. Since S is a periodic ring, $f(p) + j$ is a periodic element in S. This implies $(p, f(p) + j)$ is periodic element in $R \times S$. Consequently, $R \bowtie^f J$ is semi nil-clean.

$\square$

Theorem 2.18. *Let $f : R \to S$ be a ring homomorphism J be an ideal of S satisfying the periodic like property (i.e., for each $x \in J$ there exist distinct positive integers n and m such that $x^m = x^n$).Then $R \bowtie^f J$ is semi nil-clean if and only if R is semi nil-clean.*

Proof. ($\Rightarrow$) By Proposition 2.4(1), it is clear.
($\Leftarrow$)By [13, Theorem 2.6], $R \bowtie^f J$ is a periodic ring. Hence, $R \bowtie^f J$ is semi nil-clean.

$\square$

3. Semiclean Ring

In this section we give some characterizations of semiclean rings.

Proposition 3.1. *Let $f : R \to S$ be a ring homomorphism and J an ideal of S. If $R \bowtie^f J$ is a semiclean (resp., uniquely semiclean) ring then R is a semiclean (resp., uniquely semiclean) ring and $f(R) + J$ is a semiclean ring.*

Proof. R and $f(R)+J$ are proper homomorphic images of $R\bowtie^f J$. Then, they are semiclean.

$\square$

Theorem 3.2. *Let $f : R \to S$ be a ring homomorphism and J an ideal of S. Assume that $\dfrac{f(R)+J}{J}$ is uniquely semiclean. Then $R\bowtie^f J$ is a semiclean ring if and only if R and $f(R)+J$ are semiclean rings.*

Proof. If $R\bowtie^f J$ is a semiclean ring, then so R and $f(R)+J$ by Proposition 3.1. Conversely, assume that R and $f(R)+J$ are semiclean rings. Let $(r, j)\in R\times J$. Since R is semiclean, we can write $r = u+p$, where $u \in U(R)$ and $p\in Per(R)$. On the other hand, $f(R)+J$ is semiclean implies $f(r)+j = (f(r_1)+j_i)+(f(r_2)+j_2)$, where $(f(r_1) + j_1) \in U(f(R) + J)$ and $f(r_2) + j_2 \in Per(f(R) + J)$. It is clear that $\overline{f(r_1)} = \overline{f(r_1) + j_1}$ (resp., $\overline{f(u)}$) and $\overline{f(r_2)} = \overline{f(r_2) + j_2}$ (resp. $\overline{f(p)}$) are respectively unit and periodic element of $\dfrac{f(R)+J}{J}$, and we have $\overline{f(r)} = \overline{f(u)} + \overline{f(p)} = \overline{f(r_1)} + \overline{f(r_2)}$. Thus, $\overline{f(u)} = \overline{f(r_1)}$ and $\overline{f(p)} = \overline{f(r_2)}$ since $\dfrac{f(R)+J}{J}$ is uniquely semiclean. Therefore $f(r_1)$ and $f(r_2)$ can be written as $f(u) + j'$ and $f(p) + j''$,where j' and j'' are in J respectively. We have, $(r, f(r) + j) = (u, f(u) + j' + j_1) + (p, f(p) + j'' + j_2)$, and it is clear that $(p, f(p) + j'' + j_2)$ is an periodic element of $R\bowtie^f J$. Now it is enough to prove that $(u, f(u)+j'+j_1)$ is a unit in $R\bowtie^f J$. Since $f(u)+j_1'+j_1$ is invertible in $f(R)+J$, there exists an element $f(\alpha) + j_0$ such that $(f(u) + j_1' + j_1)(f(\alpha) + j_0) = 1$. Thus, $\overline{f(u)f(\alpha)} = \overline{1}$. Then, $\overline{f(\alpha)} = \overline{f(u^{-1})}$. So, there exists $j_0' \in J$ such that $f(\alpha) = f(u^{-1}) + j_0'$. Hence, $(u, f(u)+j_1'+j_1)(u^{-1}, f(u^{-1})+j_0'+j_0) = (u, f(u)+j_1'+j_1)(u^{-1}, f(\alpha)+j_0) = (1, 1)$. Accordingly, $(u, f(u) + j_1' + j_1)$ is invertible in $R\bowtie^f J$. Consequently, $R\bowtie^f J$ is semi clean. $\square$

Corollary 3.3. *Let $f : R \to S$ be a ring homomorphism and J an ideal of S. Assume that $\dfrac{f(R)+J}{J}$ is uniquely semiclean. Then $R\bowtie^f J$ is a semiclean ring if and only if R is a semiclean ring.*

Proof. ($\Rightarrow$) Assume that $R\bowtie^f J$ is a semiclean ring, then R is a semiclean ring since it is a homomorphic image of $R\bowtie^f J$.
($\Leftarrow$) Assume that R is a semiclean ring and let $(r, f(r) + j) \in R\bowtie^f J$. Since $r \in R$, so r can be written as $r = u + p$, where $u \in U(R)$ and $p \in Per(R)$. Then $(r, f(r)+j) = (u, f(u)+j)+(p, f(p))$. As we proved in Theorem 3.2, $(u, f(u)+j)$ is unit in $R\bowtie^f J$ and it is easy to see that $(p, f(p))$ is periodic in $R\bowtie^f J$.

$\square$

Corollary 3.4. *Let $f : R \to S$ be a ring homomorphism and J an ideal of S. Assume that $\dfrac{f(R)+J}{J}$ is uniquely semiclean and R and S are semiclean rings. Then $R\bowtie^f J$ is a semiclean ring.*

Corollary 3.5. *Let R be a ring and I an ideal such that R/I is uniquely clean. Then, $R\bowtie I$ is clean if and only if R is clean.*

Remark 3.6. Let $f : R \to S$ be a ring homomorphism and J an ideal of S.

(1) If $S = J$ then, $R \bowtie^f S$ is semiclean if and only if R and S are semiclean since $R \bowtie^f S = R \times S$.
(2) If $f^{-1}(J) = \{0\}$ then, $R \bowtie^f J$ is semiclean if and only if $f(R) + J$ is semiclean

Lemma 3.7. *Let R be a ring. Then Jacobson radical $Jac(R)$ is a semiclean ideal in R.*

Proof. If $j \in Jac(R)$, then $1 - j = u$ for some $u \in U(R)$. □

Theorem 3.8. *Let $f : R \to S$ be a ring homomorphism and J an ideal of S such that $f(u) + j$ is invertible (in S) for each $u \in U(R)$ and $j \in J$. Then, $R \bowtie^f J$ is semiclean if and only if R is semiclean.*

Proof. ($\Rightarrow$) By Proposition 3.1, R is semiclean.
($\Leftarrow$) suppose R is semiclean and $f(u) + j$ is inverible (inS) for each $u \in U(R)$ and $j \in J$. Let $(r, f(r) + j) \in R \bowtie^f J$. Since R is semiclean, $r = u + p$ where u and p are unit and periodic elements of R, respectively. Therefore $(r, f(r) + j = (u, f(u) + j) + (p, f(p))$. Now it is enough to prove that $(u, f(u) + j)$ is in $U(R \bowtie^f J)$. Take $(u^{-1}, f(u^{-1}) - jf(u^{-1})(f(u) + j)^{-1}) \in R \bowtie^f J$. Then $(u, f(u) + j)(u^{-1}, f(u^{-1}) - jf(u^{-1})(f(u) + j)^{-1}) = (1, 1)$. Hence, $R \bowtie^f J$ is semiclean. □

Theorem 3.9. *Let $f : R \to S$ be a ring homomorphism and J an ideal of S such that $J^2 = 0$. Then, $R \bowtie^f J$ is semiclean if and only if R is semiclean.*

Proof. ($\Rightarrow$) By Proposition 3.1, R is semiclean.
($\Leftarrow$) Assume that R is semiclean. Let $(r, f(r) + j) \in R \bowtie^f J$. Hence there exist an unit u and a periodic p such that $r = u + p$. Therefore $(r, f(r) + j) = (u, f(u) + j) + (p, f(p))$. It is clear that $(p, f(p))$ is periodic. Also, $(u, f(u) + j)(u^{-1}, f(u^{-1}) - f(u^{-1})^2 j) = (1, 1)$. Hence, $R \bowtie^f J$ is semiclean. □

Theorem 3.10. *Let $f : R \to S$ be a ring homomorphism and J an ideal of S such that $J \subset Nil(S)$. Then, $R \bowtie^f J$ is semiclean if and only if R is semiclean.*

Proof. $\Rightarrow$ By Proposition 3.1, R is semiclean.
($\Leftarrow$) Assume R is a semiclean ring. Let $(r, f(r)+j) \in R \bowtie^f J$. Then $(r, f(r)+j = (u, f(u)+j)+(p, f(p))$. Now it is enough to prove that $(u, f(u)+j)$ is in $U(R \bowtie^f J)$. Since $J \subset Nil(S)$, there exist k and $n \in \mathbb{N}$ such that $j^k = 0$ and $k < 2^n$. Take $(u^{-1}, f(u^{-1}) - f(u^{-1})^{2^n+1} j(f(u) - j)(f(u)^2 + j^2) \ldots (f(u)^{2^{n-1}} + j^{2^{n-1}}) \in R \bowtie^f J$. Then, $(u, f(u)+j)(u^{-1}, f(u^{-1}) - f(u^{-1})^{2^n+1} j(f(u) - j)(f(u)^2 + j^2) \ldots (f(u)^{2^{n-1}} + j^{2^{n-1}}) = (1, 1)$. Hence, $R \bowtie^f J$ is semiclean. □

Corollary 3.11. *Let R be a commutative ring and I be an ideal of R such that $I \subset Nilp(R)$. Then $R \bowtie I$ is semiclean if and only if R is semiclean.*

Theorem 3.12. *Let $f : R \to S$ be a ring homomorphism J be an ideal of S satisfying the periodic like property (i.e., for each $x \in J$ there exist distinct positive integers n and m such that $x^m = x^n$). Then $R \bowtie^f J$ is semiclean if and only if R is semiclean.*

Proof. ($\Rightarrow$) By Proposition 3.1, R is semiclean.
($\Leftarrow$)By [13, Theorem 2.6] $R \bowtie^f J$ is a periodic ring. Then by [16, Lemma 5.1] $R \bowtie^f J$ is clean ring. This gives the desired result. □

Theorem 3.13. *Let $f : R \to S$ be a ring homomorphism and J an ideal of S such that $Per(R \bowtie^f J) = \{(p, f(p)) \mid p \in Per(R)\}$. Then the following are equivalent:*

(1) *$R \bowtie^f J$ is semiclean;*
(2) *R is semiclean and $J \subseteq Jac(S)$.*

Proof. (1) $\Rightarrow$ (2) We need to prove that $J \subset Jac(S)$. Let $0 \in R$ can be write as $0 = u + p$, where $u \in U(R)$ and $p \in Per(R)$. By [16, Lemma 5.2], $p^{n(m-n)}$ is idempotent for some $m, n \in \mathbb{N}$ such that $m > n$. Therefore, $u^{n(m-n)}$ is either 1 or -1. Let $j \in J$ and $r \in S$. Choose $r' = \frac{r}{f(u^{n(m-n)-1})}$. Then $(0, r'j) = (u, f(u) + r'j) + (p, f(p))$. This implies $f(u) + r'j$ is unit. Therefore, $1 + rj = (f(u) + r'j)(f(u)^{n(m-n)-1})$ is unit.

(2) $\Rightarrow$ (1) We assume that $J \subset Rad(S)$, it is easy to see $f(u) + j = f(u)(1 + f(u^{-1}j))$ is unit in S. Then, by Theorem 3.8, $R \bowtie^f J$ is a semiclean ring. □

Corollary 3.14. *Let $f : R \to S$ be a ring homomorphism and J an ideal of S such that $J \subseteq Jac(S)$. Then $R \bowtie^f J$ is semiclean if and only if R is semiclean .*

Example 3.15. Let T be a ring, J an ideal of T, and let D be a subring of T such that $J \cap D = (0)$. If $J \subset Nilp(T)$, then the ring $D + J$ is semiclean if and only if D is semiclean.

Proof. By [2, Proposition 5.1 (3)], $D + J$ is isomorphic to the ring $D \bowtie^i J$ where $i : D \hookrightarrow J$ is the natural embedding. Thus, by Theorem 3.13, $D + J$ is semiclean if and only if D is semiclean. □

Theorem 3.16. *Let $f : R \to S$ be a ring homomorphism and J be an ideal of S such that $\dfrac{f(R) + J}{J}$ is uniquely semiclean ring. Then the following are equivalent:*

(1) *$R \bowtie^f J$ is semiclean;*
(2) *Any proper homomorphic image of $R \bowtie^f J$ is semiclean.*

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[1] Department of Mathematics, Periyar University, Tamilnadu, India.
Email address: vvijayanandmath@gmail.com

[2] Department of Mathematics, Periyar University, Tamilnadu, India.
Email address: selvavlr@yahoo.com