

ON LIE IDEALS AND MULTIPLICATIVE GENERALIZED (α, β) -REVERSE DERIVATIONS OF $\star$ -PRIME RINGS

MADHU DADHWAL ¹ AND GEETA DEVI ^{2*}

ABSTRACT. In this paper, the notion of multiplicative generalized (α, β) -reverse derivations associated with (α, β) -reverse derivations of $\star$ -prime rings is characterized. The action of these derivations on $\star$ -Lie ideals of $\star$ -prime rings is also investigated. Moreover, the commutativity of $\star$ -prime rings admitting multiplicative generalized (α, β) -reverse derivations associated with (α, β) -reverse derivations satisfying certain algebraic identities on $\star$ -Lie ideals is explored.

1. INTRODUCTION

Over the past thirty years, special type of mappings of rings including derivations, generalized derivations and generalized (α, β) -derivations have been studied by various researchers to investigate the commutativity of rings (cf. [3, 4, 5, 6, 9, 10, 11, 13, 16, 18]). Afterwards, some of these results are extended by other algebraists to analyse the actions of these mappings on appropriate subsets of rings. There are numerous results regarding commutativity of $\star$ -prime rings admitting derivations, generalized derivations and generalized (α, β) -derivations in literature. The notion of multiplicative derivations was introduced by Daif [7]. Further, Daif and Tammam-El-Sayaid [8] extended the aforementioned notion to multiplicative generalized derivations which inspires us to examine the multiplicative generalized (α, β) -reverse derivations.

Recall that a ring R is prime if for any $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$. For any $a, b \in R$ the symbol $[a, b]$ (resp. $a \circ b$) represents the commutator $ab - ba$ (resp. $ab + ba$) and an additive subgroup L of R is said to be a Lie ideal of R if $[L, R] \subseteq L$. Further, an additive mapping $\star : R \rightarrow R$ is called an involution if $(ab)^\star = b^\star a^\star$ and $(a^\star)^\star = a$, for all $a, b \in R$. A ring R equipped with an involution is called a $\star$ -ring or a ring with involution and an ideal I of R is called a $\star$ -ideal if $I^\star = I$. A ring with involution is said to be a $\star$ -prime ring if $aRb = aRb^\star = (0)$ or $aRb = a^\star Rb = (0)$ implies that either $a = 0$ or $b = 0$. Every prime ring with an involution is a $\star$ -prime ring, but converse need not to be true in general and this was justified by an example in [12]. Throughout this paper,

Date: Received: Oct 1, 2022; Accepted: Feb 19, 2023.

* Corresponding author.

2010 *Mathematics Subject Classification.* 16W25; 16W10; 16N60; 16U80.

Key words and phrases. $\star$ -Prime rings, Lie ideals, (α, β) -reverse derivations, multiplicative generalized (α, β) -reverse derivations, commutativity.

the symbol $S_{a\star}(R)$ denotes the set of symmetric and skew symmetric elements of R , i.e. $S_{a\star}(R) = \{a \in R : a^\star = \pm a\}$.

Further, a Lie ideal is said to be a $\star$ -Lie ideal if $L^\star = L$ and if L is a Lie (resp. $\star$ -Lie) ideal of R , then we say that L is a 2-Lie (resp. $(\star, 2)$ -Lie) ideal of R if $2uv \in L$, for all $u, v \in L$. Also, if L is a 2-Lie (resp. $(\star, 2)$ -Lie) ideal of R and $u, v \in L, r \in R$, then $2r[u, v] = 2[ru, v] - 2[r, v]u, 2[u, v]r = 2[u, vr] - 2v[u, r]$ implies that $2R[L, L] \subseteq L$ and $2[L, L]R \subseteq L$. An additive mapping $d : R \rightarrow R$ is called a reverse derivation if $d(ab) = d(b)a + bd(a)$, for all $a, b \in R$. An additive mapping $F : R \rightarrow R$ is said to be a generalized reverse derivation if $F(ab) = F(b)a + bd(a)$, for all $a, b \in R$, where d is a reverse derivation of R . Let α and β be two endomorphisms of R . Then an additive mapping $d : R \rightarrow R$ is called an (α, β) -derivation if $d(ab) = d(a)\alpha(b) + \beta(a)d(b)$, for all $a, b \in R$. Following [1], an additive mapping $F : R \rightarrow R$ is said to be a generalized (α, β) -derivation associated with an (α, β) -derivation d of R if $F(ab) = F(a)\alpha(b) + \beta(a)d(b)$, for all $a, b \in R$. Furthermore, a mapping $F : R \rightarrow R$ (not necessarily additive) is called a multiplicative generalized derivation if it satisfies $F(ab) = F(a)b + ad(b)$, for all $a, b \in R$, where d is a derivation of R . However, a map $F : R \rightarrow R$ (not necessarily additive) is called a multiplicative generalized (α, β) -derivation if it satisfies $F(ab) = F(a)\alpha(b) + \beta(a)d(b)$, for all $a, b \in R$, where d is an (α, β) -derivation of R . An additive mapping $d : R \rightarrow R$ is called an (α, β) -reverse derivation if $d(ab) = d(b)\alpha(a) + \beta(b)d(a)$, for all $a, b \in R$. Moreover, a map $F : R \rightarrow R$ (not necessarily additive) is said to be a multiplicative generalized (α, β) -reverse derivation if $F(ab) = F(b)\alpha(a) + \beta(b)d(a)$, for all $a, b \in R$, where d is an (α, β) -reverse derivation of R .

In [17], Rehman and Golbaşı had studied generalized (α, β) -derivations of $\star$ -prime rings and proved some results for generalized (α, β) -derivations and $\star$ -Lie ideals of $\star$ -prime rings. This study stimulates us to investigate the concept of multiplicative generalized (α, β) -reverse derivations of $\star$ -prime rings.

In Section 2, we list some basic results including commutator identities, Jordan identities and $\star$ -Lie ideals which are very useful in proving the main theorems of this paper. The last section deals with the behaviour of multiplicative generalized (α, β) -reverse derivations on $\star$ -Lie ideals of $\star$ -prime rings. In this direction, we present some conditions satisfied by multiplicative generalized (α, β) -reverse derivations of $\star$ -prime rings on $\star$ -Lie ideals which demonstrate that $L \subseteq Z(R)$. Throughout this paper, $Z(R)$ denotes the center of an associative ring R and $\star$ is an involution of R .

2. SOME BASIC RESULTS

First, we turn our attention to collect some examples and prove some results which we will use frequently in the sequel. It is also observed that $(\star, 2)$ Lie ideals are more general than the ideals of R . We begin with an example, which shows that every $(\star, 2)$ -Lie ideal of R need not be an ideal of R .

Example 2.1. Let R be the ring of integers and take $R_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in R \right\}$. Clearly, R_1 is a ring under the usual addition and multiplication of matrices.

Define map $\star : R_1 \rightarrow R_1$ by $\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)^\star = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Then it is easy to check that $\star$ is an involution. Also, $L = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in R \right\}$ is a $\star$ as well as a 2 - Lie ideal of R_1 i.e., $(\star, 2)$ - Lie ideal, which is not an ideal.

Example 2.2. Let R be a commutative ring and take $R_1 = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} : a, b \in R \right\}$. Obviously, R_1 is a ring under the usual addition and multiplication of matrices.

Define maps $\star, F, d, \alpha, \beta : R_1 \rightarrow R_1$ by $\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} \right)^\star = \begin{bmatrix} 0 & -a & b \\ 0 & 0 & -a \\ 0 & 0 & 0 \end{bmatrix}$, $F\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & a & a^2b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $d\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\alpha\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\beta\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & -a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Clearly, $\star$ is an involution and F is multiplicative generalized (α, β) -reverse derivation associated with (α, β) -reverse derivation d of R_1 .

The proof of the upcoming lemma is quite easy so we omit the proof.

Lemma 2.3. *If R is a ring and $a, b, c \in R$, then the following statements hold:*

- (i) $[a, bc] = b[a, c] + [a, b]c$; (ii) $[ab, c] = a[b, c] + [a, c]b$; (iii) $[a, b+c] = [a, b] + [a, c]$;
- (iv) $[a+b, c] = [a, c] + [b, c]$; (v) $[ab, a] = a[b, a]$; (vi) $[a, ab] = a[a, b]$; (vii) $[a, ba] = [a, b]a$;
- (viii) $[ba, a] = [b, a]a$; (ix) $a \circ (bc) = (a \circ b)c - b[a, c] = b(a \circ c) + [a, b]c$;
- (x) $(ab) \circ c = a(b \circ c) - [a, c]b = (a \circ c)b + a[b, c]$.

Lemma 2.4. [15, Lemma 4] *Let R be a $\star$ -prime ring with $\text{char}(R) \neq 2$, L be a nonzero $\star$ -Lie ideal of R and $a, b \in R$. If $aLb = \star(a)Lb = (0)$, then $a = 0$ or $b = 0$ or $L \subseteq Z(R)$.*

Lemma 2.5. [14, Lemma 2.3] *Let R be a $\star$ -prime ring with $\text{char}(R) \neq 2$ and L a nonzero $\star$ -Lie ideal of R . If $[L, L] \subseteq Z(R)$, then $L \subseteq Z(R)$.*

Lemma 2.6. *Let R be a $\star$ -prime ring with $\text{char}(R) \neq 2$ and L be a nonzero $\star$ -Lie ideal of R . Suppose that there exists a reverse (α, β) -derivation d with α, β automorphisms of R such that $d(L) = (0)$, then either $L \subseteq Z(R)$ or $d = 0$.*

Proof. By the given hypothesis, we have $d(u) = 0$, for all $u \in L$. By replacing u by $[u, r]$, we get

$$d(r)\alpha(u) - \beta(u)d(r) = 0 \quad (2.1)$$

for all $u \in L$ and $r \in R$. Taking rv in place of r in (2.1), we find that

$$(d(v)\alpha(r) + \beta(v)d(r))\alpha(u) - \beta(u)(d(v)\alpha(r) + \beta(v)d(r)) = 0$$

for all $u, v \in L$ and $r \in R$. By using hypothesis, the above equation infers that

$$\beta(v)d(r)\alpha(u) - \beta(u)\beta(v)d(r) = 0. \quad (2.2)$$

From (2.1), we have

$$d(r)\alpha(u) = \beta(u)d(r). \quad (2.3)$$

Combining (2.2) and (2.3), we obtain $0 = \beta(v)\beta(u)d(r) - \beta(u)\beta(v)d(r) = [\beta(v), \beta(u)]d(r)$, for all $u, v \in L$ and $r \in R$. Further, putting rs instead of r , the last relation infers that

$$[\beta(v), \beta(u)]\beta(s)d(r) = 0$$

for all $u, v \in L$ and $r, s \in R$. Since β is an automorphism, so the above equation infers that $[v, u]R\beta^{-1}(d(r)) = (0)$, for all $u, v \in L$ and $r \in R$ and also L is a $\star$ -Lie ideal, so

$$([v, u])^*R\beta^{-1}(d(r)) = (0)$$

As R is a $\star$ -prime ring, so $[v, u]R\beta^{-1}(d(r)) = ([v, u])^*R\beta^{-1}(d(r)) = (0)$, for all $u, v \in L$ and $r \in R$ implies that either $[L, L] = (0)$ or $d = 0$. By using Lemma 2.5, the former case implies that $L \subseteq Z(R)$. $\square$

Lemma 2.7. *If L is a $(\star, 2)$ -Lie ideal of a $\star$ -prime ring R with $\text{char}(R) \neq 2$ and $L \not\subseteq Z(R)$, then there exists a nonzero $\star$ -ideal $I = R[L, L]R$ of R generated by $[L, L]$ such that $2I \subseteq L$.*

Proof. Let L be a $(\star, 2)$ -Lie ideal of a $\star$ -prime ring R with $\text{char}(R) \neq 2$ and $L \not\subseteq Z(R)$. Clearly, $L \not\subseteq Z(R)$ infers that L is nonzero. First, we will show that $[L, L] \neq (0)$. If possible, let $[L, L] = (0)$. Then, by Lemma 2.5, $L \subseteq Z(R)$, a contradiction. Thus, $[L, L] \neq (0)$. Let $I = R[L, L]R$ be a $\star$ -ideal of R generated by $[L, L]$. Since L is a $(\star, 2)$ -Lie ideal of R , so $2r[u, v] = 2[ru, v] - 2[r, v]u \in L$, for all $u, v \in L$ and $r \in R$. This infers that $2r[u, v]s = 2[r[u, v], s] + 2sr[u, v] \in L$, for all $u, v \in L$ and $r, s \in R$. Hence, $2I = 2R[L, L]R \subseteq L$. If $I = (0)$, then we have $s[u, v]R = 0$, which infers that $s[u, v]Rl = (0) = s[u, v]R(l)^*$, for all $l, u, v \in L, s \in R$. By the $\star$ -primeness of R , get $R[u, v] = (0)$. This infers that $wR[u, v] = (0) = wR([u, v])^*$, for all $u, v, w \in L$ and again by the $\star$ -primeness of R , we get $[L, L] = (0)$, as L is nonzero $\star$ -Lie ideal. Moreover, by Lemma 2.5, $L \subseteq Z(R)$, which is a contradiction. Thus I is nonzero $\star$ -ideal of R . $\square$

Lemma 2.8. [2, Lemma 2.7] *Let R be a $\star$ -prime ring with $\text{char}(R) \neq 2$ and L be a nonzero $\star$ -Lie ideal of R . If $a \in R$ such that $[a, L] \subseteq Z(R)$, then either $L \subseteq Z(R)$ or $a \in Z(R)$.*

Lemma 2.9. *Let R be a $\star$ -prime ring with $\text{char}(R) \neq 2$ and L a nonzero $\star$ -Lie ideal of R . If β and α are automorphisms of R such that $[\beta(L), \alpha(L)] = (0)$, then $L \subseteq Z(R)$.*

Proof. By the given hypothesis, we have $[\beta(u), \alpha(v)] = 0$, for all $u, v \in L$. This infers that $0 = [\beta(u), \alpha([v, r])] = [\beta(u), \alpha(v)\alpha(r) - \alpha(r)\alpha(v)]$, for all $u, v \in L$ and

$r \in R$. By using commutator identities and hypothesis in the last equation, we get

$$0 = \alpha(v)[\beta(u), \alpha(r)] - [\beta(u), \alpha(r)]\alpha(v) = [\alpha(v), [\beta(u), \alpha(r)]]$$

for all $u, v \in L$ and $r \in R$. Further, replacing r by rj , we get $0 = [\beta(u), \alpha(r)][\alpha(v), \alpha(j)] + [\alpha(v), \alpha(r)[\beta(u), \alpha(j)]]$, for all $u, v, j \in L$ and $r \in R$ and by using hypothesis,

$$[\beta(u), \alpha(r)][\alpha(v), \alpha(j)] = 0.$$

On taking sr instead of r , the above equation gives $[\beta(u), \alpha(s)]\alpha(r)[\alpha(v), \alpha(j)] = 0$, for all $u, v, j \in L$ and $r, s \in R$. As α is an automorphism, so

$$\alpha^{-1}([\beta(u), \alpha(s)])R[v, j] = (0)$$

for all $u, v, j \in L$ and $s \in R$. This implies that

$$\alpha^{-1}([\beta(u), \alpha(s)])R([v, j])^* = (0)$$

for all $u, v, j \in L$ and $s \in R$, as L is a $\star$ -Lie ideal and by the $\star$ -primeness of R , we obtain that either $[\beta(L), R] = (0)$ or $[L, L] = (0)$. The former case implies that $\beta([L, R]) = (0)$, as β is an automorphism. This gives that $L \subseteq Z(R)$. By Lemma 2.5, the latter case $[L, L] = (0)$ also infers that $L \subseteq Z(R)$. Hence $L \subseteq Z(R)$. $\square$

Proposition 2.10. *Let F be a multiplicative generalized (α, β) -reverse derivation of a $\star$ -prime ring R associated with an (α, β) -reverse derivation d , $\text{char}(R) \neq 2$ and L be a $(\star, 2)$ -Lie ideal of R . If α and β are automorphisms such that $F([L, L]) = (0)$, then either $L \subseteq Z(R)$ or $d = 0$.*

Proof. By the given hypothesis, we have

$$F([u, v]) = 0 \tag{2.4}$$

for all $u, v \in L$. If $L \subseteq Z(R)$, then there is nothing to prove. Suppose that $L \not\subseteq Z(R)$. On replacing v by $2uv$ in (2.4), we get $0 = 2F(u[u, v]) = 2(F([u, v])\alpha(u) + \beta([u, v])d(u))$. By using the fact that $\text{char}(R) \neq 2$ and (2.4), the previous equation infers that

$$\beta([u, v])d(u) = 0$$

for all $u, v \in L$. Now, taking $2vk$ in place of v and using $\text{char}(R) \neq 2$, the preceding equation implies that $\beta([u, v])\beta(k)d(u) = 0$, for all $u, v, k \in L$. As β is an automorphism, so $[u, v]L\beta^{-1}(d(u)) = (0)$, for all $u, v \in L$. Since L is a $\star$ -Lie ideal of R which yields that

$$([u, v])^*L\beta^{-1}(d(u)) = (0)$$

for all $v \in L$, $u \in L \cap S_{a\star}(R)$. By Lemma 2.4, for each $u \in L \cap S_{a\star}(R)$, either $[u, L] = (0)$ or $d(u) = 0$. Let $u \in L$, as $u + u^*, u - u^* \in L \cap S_{a\star}(R)$ and $[u \pm u^*, L] = (0)$ or $d(u \pm u^*) = 0$. Hence we have for each $u \in L$, either $[u, L] = (0)$ or $d(u) = 0$. Let $A = \{u \in L : [u, L] = (0)\}$ and $B = \{u \in L : d(u) = 0\}$. Therefore, A and B are additive subgroups of L and $L = A \cup B$. In view of Brauer's trick it follows that either $L = A$ or $L = B$.

If $L = A$, then $[L, L] = (0)$ then by Lemma 2.5, $L \subseteq Z(R)$, which is not possible. In the latter case, we have $d(L) = (0)$. As $L \not\subseteq Z(R)$, so by Lemma 2.6, $d = 0$. With this our proof is complete. $\square$

Corollary 2.11. *Let F be a multiplicative generalized (α, β) -reverse derivation of a $\star$ -prime R associated with an (α, β) -reverse derivation d , $\text{char}(R) \neq 2$. If α and β are automorphisms such that $F([R, R]) = (0)$, then either R is commutative or $d = 0$.*

In the upcoming section, R represents a $\star$ -prime ring with $\text{char}(R) \neq 2$, L is a nonzero $(\star, 2)$ -Lie ideal and α, β are automorphisms of R .

3. MAIN RESULTS

In [17] Rehman proved some results regarding generalized (α, β) -derivations of $\star$ -prime rings which motivates us to study and investigate generalized (α, β) -reverse derivations of $\star$ -prime rings. Moreover, we investigate the actions of these derivations in the framework of $\star$ -Lie ideals.

Theorem 3.1. *Let F be a multiplicative generalized (α, β) -reverse derivation of R associated with an (α, β) -reverse derivation d and $0 \neq a \in R$. If any one of the following holds true:*

$$(i) \ a(F(uv) + \alpha(vu)) = 0$$

$$(ii) \ a(F(uv) - \alpha(vu)) = 0$$

for all $u, v \in L$. Then $L \subseteq Z(R)$.

Proof. (i) By given hypothesis, we have $0 = a(F(uv) + \alpha(vu)) = a(F(v)\alpha(u) + \beta(v)d(u) + \alpha(vu))$, for all $u, v \in L$. If possible, suppose that $L \not\subseteq Z(R)$. Putting $2vi$ in place of v in the hypothesis and using $\text{char}(R) \neq 2$, we have

$$a(F(vi)\alpha(u) + \beta(v)\beta(i)d(u) + \alpha(v)\alpha(i)\alpha(u)) = 0 \quad (3.1)$$

for all $u, v, i \in L$. From hypothesis, we have

$$aF(vi) = -a\alpha(iv). \quad (3.2)$$

On combining (3.1) and (3.2), we find that

$$a([\alpha(v), \alpha(i)]\alpha(u) + \beta(v)\beta(i)d(u)) = 0 \quad (3.3)$$

for all $u, v, i \in L$. Putting $2uk$ instead of u in last relation, we get

$$a(2[\alpha(v), \alpha(i)]\alpha(u)\alpha(k) + 2\beta(v)\beta(i)(d(k)\alpha(u) + \beta(k)d(u))) = 0 \quad (3.4)$$

for all $u, v, i, k \in L$. From (3.3), we obtain that $a\beta(v)\beta(i)d(k) = -a[\alpha(v), \alpha(i)]\alpha(k)$, $a\beta(v)\beta(2ik)d(u) = -a[\alpha(v), \alpha(2ik)]\alpha(u)$, as $2ik \in L$. On using the last relations in (3.4),

$$2a([\alpha(v), \alpha(i)]\alpha(u)\alpha(k) - [\alpha(v), \alpha(i)]\alpha(k)\alpha(u) - [\alpha(v), \alpha(ik)]\alpha(u)) = 0.$$

Since $\text{char}(R) \neq 2$, so

$$a([\alpha(v), \alpha(i)][\alpha(u), \alpha(k)] - [\alpha(v), \alpha(ik)]\alpha(u)) = 0 \quad (3.5)$$

for all $u, v, i, k \in L$. Further, replacing u by $2ut$ in (3.5), we conclude that

$$a([\alpha(v), \alpha(i)]\alpha(u)[\alpha(t), \alpha(k)] = 0$$

for all $u, v, i, k, t \in L$. Since α is an automorphism, therefore $\alpha^{-1}(a)[v, i]L[t, k] = (0)$, for all $v, i, k, t \in L$. As L is a $\star$ -Lie ideal of R , therefore

$$\alpha^{-1}(a)[v, i]L([t, k])^\star = (0)$$

for all $v, i, k, t \in L$. By Lemma 2.4, the preceding equation implies that either $\alpha^{-1}(a)[v, i] = 0$, for all $v, i \in L$ or $[L, L] = (0)$. By Lemma 2.5, the latter case infers that $L \subseteq Z(R)$, which is a contradiction. From the former case, we have $\alpha^{-1}(a)[v, i] = 0$, for all $v, i \in L$. On taking $2vu$ in place of v , we are left with $\alpha^{-1}(a)L[u, i] = (0)$, for all $u, i \in L$. This yields $\alpha^{-1}(a)L([u, i])^* = (0)$, as L is a $\star$ -Lie ideal of R and by Lemma 2.4, $[L, L] = (0)$, as $a \neq 0$. Again by Lemma 2.5, $L \subseteq Z(R)$, a contradiction. Therefore, in each case, we get a contradiction. Hence $L \subseteq Z(R)$.

By the same way, one can easily prove (ii). $\square$

A direct outcome of the above theorem is

Corollary 3.2. *If F is a multiplicative generalized (α, β) -reverse derivation of R associated with an (α, β) -reverse derivation d and $0 \neq a \in R$ such that $a(F(uv) \pm \alpha(vu)) = 0$, for all $u, v \in R$, then R is commutative.*

Theorem 3.3. *Let F be a multiplicative generalized (α, β) -reverse derivation of R associated with a nonzero (α, β) -reverse derivation d . If any one of the following holds true:*

- (i) $F(uv) + \beta(u)F(v) = 0$
- (ii) $F(uv) - \beta(u)F(v) = 0$ for all $u, v \in L$. Then $L \subseteq Z(R)$.

Proof. (i) By given hypothesis, we have $F(uv) + \beta(u)F(v) = 0$, for all $u, v \in L$. Assume that $L \not\subseteq Z(R)$. Now, taking $2ui$ in place of u in hypothesis, we find $0 = 2(F(uiv) + \beta(ui)F(v)) = 2(F(iv)\alpha(u) + \beta(iv)d(u) + \beta(u)\beta(i)F(v))$, for all $u, v, i \in L$. As $\text{char}(R) \neq 2$, so the previous equation infers that

$$F(iv)\alpha(u) + \beta(iv)d(u) + \beta(u)\beta(i)F(v) = 0. \quad (3.6)$$

By using the hypothesis, we have $F(iv) = -\beta(i)F(v)$. Combining the last relation with equation (3.6), we get

$$-\beta(i)F(v)\alpha(u) + \beta(i)\beta(v)d(u) + \beta(u)\beta(i)F(v) = 0 \quad (3.7)$$

for all $u, v, i \in L$. Putting $2zi$ instead of i , we obtain

$$\beta(z)(-\beta(i)F(v)\alpha(u) + \beta(i)\beta(v)d(u)) + \beta(u)\beta(z)\beta(i)F(v) = 0 \quad (3.8)$$

for all $u, v, i, z \in L$. From (3.7),

$$-\beta(i)F(v)\alpha(u) + \beta(i)\beta(v)d(u) = -\beta(u)\beta(i)F(v)$$

and using this in (3.8), we get $[\beta(u), \beta(z)]\beta(L)F(v) = (0)$, for all $u, v, z \in L$. As β is an automorphism, so $[u, z]L\beta^{-1}(F(v)) = (0)$, for all $u, v, z \in L$. From this we have

$$([u, z])^*L\beta^{-1}(F(v)) = (0)$$

for all $u, v, z \in L$. By Lemma 2.4, we conclude that, either $[L, L] = (0)$ or $F(L) = (0)$. From the former case, we have $[L, L] = (0)$. In view of Lemma 2.5, it follows that $L \subseteq Z(R)$, which is a contradiction to our assumption. The latter case implies that $F([L, L]) = (0)$. Since d is nonzero, therefore by Proposition 2.10, $L \subseteq Z(R)$, a contradiction. Hence $L \subseteq Z(R)$.

(ii) can be proved by using the same techniques as in (i). $\square$

Corollary 3.4. *Let F be a multiplicative generalized (α, β) -reverse derivation of R associated with a nonzero (α, β) -reverse derivation d such that $F(uv) \pm \beta(u)F(v) = 0$, for all $u, v \in R$. Then R is commutative.*

Corollary 3.5. *Let F be a multiplicative generalized reverse derivation of R associated with a nonzero reverse derivation d such that $F(uv) \pm uF(v) = 0$, for all $u, v \in L$. Then $L \subseteq Z(R)$.*

Proposition 3.6. *Let F be a multiplicative generalized (α, β) -reverse derivation of R associated with a nonzero (α, β) -reverse derivation d and $0 \neq a \in R$. If β, d commute with $\star$ such that $a(F(uv) + F(v)F(u)) = 0$, for all $u, v \in L$. Then $L \subseteq Z(R)$.*

Proof. By given hypothesis, we have

$$a(F(uv) + F(v)F(u)) = 0 \quad (3.9)$$

for all $u, v \in L$. Suppose on contrary that $L \not\subseteq Z(R)$. Putting $2ui$ in place of u in (3.9), we get

$$\begin{aligned} 0 &= 2a(F(uiv) + F(v)F(ui)) \\ &= 2a(F(iv)\alpha(u) + \beta(iv)d(u) + F(v)F(i)\alpha(u) + F(v)\beta(i)d(u)) \\ &= 2a((F(iv) + F(v)F(i))\alpha(u) + \beta(iv)d(u) + F(v)\beta(i)d(u)) \end{aligned}$$

for all $u, v, i \in L$. By using $\text{char}(R) \neq 2$ and using (3.9), we get that $a(\beta(iv)d(u) + F(v)\beta(i)d(u)) = 0$. This infers that

$$a(\beta(iv) + F(v)\beta(i))d(u) = 0$$

for all $u, v, i \in L$. Replacing u by $2zu$ in the above equation, we obtain $a(\beta(iv) + F(v)\beta(i))\beta(u)d(z) = 0$, for all $u, v, i, z \in L$. This concludes that

$$\beta^{-1}(a(\beta(iv) + F(v)\beta(i)))L\beta^{-1}(d(z)) = 0$$

for all $v, i, z \in L$. From this we have $\beta^{-1}(a(\beta(iv) + F(v)\beta(i)))L(\beta^{-1}(d(z)))^\star = (0)$, for all $v, i, z \in L$, because L is a $\star$ -Lie ideal and β, d commute with $\star$. By Lemma 2.4, we find that either $a(\beta(iv) + F(v)\beta(i)) = 0$, for all $v, i \in L$ or $d(L) = (0)$. Now, consider the case $d(L) = (0)$. As d is nonzero, thus by Lemma 2.6, $L \subseteq Z(R)$, a contradiction to our assumption. The former case implies that

$$a(\beta(iv) + F(v)\beta(i)) = 0 \quad (3.10)$$

for all $v, i \in L$. Taking $2ij$ in place of i in (3.10), we are left with

$$a(\beta(i)\beta(j)\beta(v) + F(v)\beta(i)\beta(j)) = 0$$

for all $i, j, v \in L$. From (3.10), $aF(v)\beta(i) = -a\beta(iv)$ and using this in the above equation, we conclude that $a\beta(i)\beta([j, v]) = 0$ and $\beta^{-1}(a)L[j, v] = (0)$, for all $j, v \in L$. This also gives $\beta^{-1}(a)L([j, v])^\star = (0)$, for all $j, v \in L$, as $L = L^\star$. Since $a \neq 0$, so by Lemma 2.4 and Lemma 2.5, we get $L \subseteq Z(R)$, again a contradiction. Hence $L \subseteq Z(R)$. $\square$

Corollary 3.7. *Let F be a multiplicative generalized reverse derivation of R associated with a nonzero reverse derivation d and $0 \neq a \in R$. If d commutes with $\star$ such that $a(F(uv) + F(v)F(u)) = 0$, for all $u, v \in L$. Then $L \subseteq Z(R)$.*

Theorem 3.8. *Let F be a multiplicative generalized (α, β) -reverse derivation of R associated with an (α, β) -reverse derivation d , $0 \neq a \in S_{a\star}(R)$ and α commutes with $\star$ such that $a(F(u)F(v) + \alpha(uv)) = 0$, for all $u, v \in L$. Then $L \subseteq Z(R)$.*

Proof. By given hypothesis, we have

$$a(F(u)F(v) + \alpha(uv)) = 0 \quad (3.11)$$

for all $u, v \in L$. Replacing $2vi$ in place of v , we find

$$0 = a(F(u)F(vi) + \alpha(uvi)) = 2a(F(u)F(i)\alpha(v) + F(u)\beta(i)d(v) + \alpha(uvi))$$

for all $u, v, i \in L$. In view of (3.11), $aF(u)F(i) = -a\alpha(u)\alpha(i)$, so the last equation infers that

$$a(\alpha(u)[\alpha(v), \alpha(i)] + F(u)\beta(i)d(v)) = 0 \quad (3.12)$$

for all $u, v, i \in L$. Putting $2vk$ in place of v in (3.12), we get

$$a(2\alpha(u)[\alpha(v), \alpha(i)]\alpha(k) + 2\alpha(u)\alpha(v)[\alpha(k), \alpha(i)] + 2F(u)\beta(i)d(k)\alpha(v) + F(u)\beta(2ik)d(v)) = 0 \quad (3.13)$$

for all $u, v, i, k \in L$. By using (3.12), we get

$$aF(u)\beta(i)d(k) = -a\alpha(u)[\alpha(k), \alpha(i)], aF(u)\beta(2ik)d(v) = -a\alpha(u)[\alpha(v), \alpha(2ik)]$$

as $2ik \in L$. On using the above relation in (3.13) and $\text{char}(R) \neq 2$, we obtain

$$a\alpha(u)([\alpha(v), \alpha(i)]\alpha(k) + \alpha(v)[\alpha(k), \alpha(i)] - [\alpha(k), \alpha(i)]\alpha(v) - [\alpha(v), \alpha(ik)]) = 0$$

for all $u, v, i, k \in L$. This implies that

$$\alpha^{-1}(a)L([v, i]k + v[k, i] - [k, i]v - [v, ik]) = (0)$$

for all $u, v, i, k \in L$. Since $a \in S_{a\star}(R)$ and α commutes with $\star$, so $(\alpha^{-1}(a))^\star L([v, i]k + v[k, i] - [k, i]v - [v, ik]) = (0)$, for all $u, v, i, k \in L$ and by Lemma 2.4, either

$$[v, i]k + v[k, i] - [k, i]v - [v, ik] = 0 \quad (3.14)$$

for all $v, i, k \in L$ or $L \subseteq Z(R)$. If equation (3.14) holds, then by taking $2kv$ in place of k in (3.14), we are left with

$$0 = vk[v, i] - k[v, i]v = [v, k[v, i]]$$

for all $i, k, v \in L$. Further, replacing k by $2kw$ in the preceding equation, we conclude that $[v, k]w[v, i] = (0)$ for all $i, k, v, w \in L$. This leads to $[v, k]L[v, i] = (0)$, for all $v, k, i \in L$. Since L is a $\star$ -Lie ideal, thus $([v, k])^\star L[v, i] = (0)$, for all $k, i \in L, v \in L \cap S_{a\star}(R)$. By Lemma 2.4, for each $v \in L \cap S_{a\star}(R)$, we conclude that $[v, L] = (0)$. Moreover, by Lemma 2.8, either $L \subseteq Z(R)$ or $L \cap S_{a\star}(R) \subseteq Z(R)$. As $u + u^\star, u - u^\star \in L \cap S_{a\star}(R)$, for all $u \in L$. Thus, $u \pm u^\star \in Z(R)$, for all $u \in L$ and it yields $L \subseteq Z(R)$. $\square$

Corollary 3.9. *If F is a multiplicative generalized (α, β) -reverse derivation of R associated with an (α, β) -reverse derivation d and $0 \neq a \in S_{a\star}(R)$ and α commutes with $\star$ such that $a(F(u)F(v) + \alpha(uv)) = 0$, for all $u, v \in R$, then R is commutative.*

Proposition 3.10. *Let F, G be two multiplicative generalized (α, β) -reverse derivations of R associated with (α, β) -reverse derivations d, g respectively and β, g commute with $\star$. If I be a nonzero $\star$ -ideal of R and g is nonzero such that $F(u)\alpha(v) + \alpha(v)G(u) = 0$, for all $u, v \in I$. Then $I \subseteq Z(R)$.*

Proof. By given hypothesis, we have

$$F(u)\alpha(v) + \alpha(v)G(u) = 0 \quad (3.15)$$

for all $u, v \in I$. Replacing u by vu in (3.15), we obtain

$$0 = F(vu)\alpha(v) + \alpha(v)G(vu) = (F(u)\alpha(v) + \alpha(v)G(u))\alpha(v) + \beta(u)d(v)\alpha(v) + \alpha(v)\beta(u)g(v).$$

Further, by using (3.15), the previous relation infers that

$$\beta(u)d(v)\alpha(v) + \alpha(v)\beta(u)g(v) = 0 \quad (3.16)$$

for all $u, v \in I$. Putting ru in place of u in (3.16), we find

$$\beta(r)\beta(u)d(v)\alpha(v) + \alpha(v)\beta(r)\beta(u)g(v) = 0 \quad (3.17)$$

for all $u, v \in I$ and $r \in R$. From (3.16), we have $\beta(u)d(v)\alpha(v) = -\alpha(v)\beta(u)g(v)$ and using this in the above equation, we conclude that $[\alpha(v), \beta(r)]\beta(u)g(v) = (0)$. This implies that

$$\beta^{-1}([\alpha(v), \beta(r)])I\beta^{-1}(g(v)) = (0)$$

for all $v \in I$ and $r \in R$. Since β, g commute with $\star$ and I is a $\star$ -ideal of R , therefore

$$\beta^{-1}([\alpha(v), \beta(r)])I(\beta^{-1}(g(v)))^{\star} = (0)$$

for all $v \in I \cap S_{a\star}(R)$ and $r \in R$. As β is an automorphism and I is a nonzero $\star$ -ideal of a $\star$ -prime ring R , therefore for each $v \in I \cap S_{a\star}(R)$, either $[\alpha(v), \beta(R)] = (0)$ or $g(v) = 0$. Take $v \in I$, as $v + (v)^{\star}, v - (v)^{\star} \in I \cap S_{a\star}(R)$ and either $[\alpha(v \pm (v)^{\star}), \beta(R)] = (0)$ or $g(v \pm (v)^{\star}) = 0$. This concludes that for each $v \in I$ either $[\alpha(v), \beta(R)] = (0)$ or $g(v) = 0$. By applying Brauer's trick, either $[\alpha(I), \beta(R)] = (0)$ or $g(I) = (0)$. From former case, we have $[\alpha(I), \beta(I)] = (0)$ and by Lemma 2.9, $I \subseteq Z(R)$. By Lemma 2.6, the latter case gives that $I \subseteq Z(R)$. $\square$

Theorem 3.11. *Let F, G be two multiplicative generalized (α, β) -reverse derivations of R associated with (α, β) -reverse derivations d, g respectively and β, g commute with $\star$. If g is nonzero such that $F(u)\alpha(v) + \alpha(v)G(u) = 0$, for all $u, v \in L$. Then $L \subseteq Z(R)$.*

Proof. By the given hypothesis, we have $F(u)\alpha(v) + \alpha(v)G(u) = 0$, for all $u, v \in L$. Assume on the contrary that $L \not\subseteq Z(R)$. By Lemma 2.7, there exists a nonzero ideal $I = R[L, L]R$ of R such that $2I \subseteq L$. In view of this, we have

$$F(u)\alpha(v) + \alpha(v)G(u) = 0$$

for all $u, v \in 2I$. In view of the above proposition, $2I \subseteq Z(R)$. This concludes that $2r[u, v]s \in Z(R)$, for all $u, v \in L$ and $r, s \in R$. By replacing s by sz , we have $2r[u, v]sz \in Z(R)$, for all $u, v, z \in L$ and $r, s \in R$. Since $2r[u, v]s \in Z(R)$, therefore $2r[u, v]s[z, w] = 0$, for all $u, v, w, z \in L$ and $r, s \in R$. This leads to $2r[u, v]R[z, w] = (0)$ and $2r[u, v]R([z, w])^{\star} = (0)$, for all $u, v, w, z \in L$ and $r, s \in R$, as L is a $\star$ -Lie ideal of R . The $\star$ -primeness of R concludes that, either

$2r[u, v] = (0)$, for all $u, v \in L$ and $r \in R$ or $[L, L] = (0)$. As $\text{char}(R) \neq 2$ and L is nonzero $\star$ -Lie ideal of R , so the former case gives that $lR[u, v] = (l)^\star R[u, v] = (0)$, for all $u, v, l \in L$. Again by the $\star$ -primeness of R , $[L, L] = (0)$. Therefore, in each case, we get $[L, L] = (0)$. In view of Lemma 2.5, $L \subseteq Z(R)$, a contradiction to our assumption. Hence $L \subseteq Z(R)$. $\square$

Theorem 3.12. *Let F, G be two multiplicative generalized (α, β) -reverse derivations of R associated with (α, β) -reverse derivations d, g respectively. If $F(uv) + G(uv) + \alpha(vu) = 0$, for all $u, v \in L$, then $L \subseteq Z(R)$.*

Proof. Let us assume that $L \not\subseteq Z(R)$. By the given hypothesis, we have $F(uv) + G(uv) + \alpha(vu) = 0$. Taking $2vi$ in place of v and using the fact that $\text{char}(R) \neq 2$, we get

$$(F(vi) + G(vi))\alpha(u) + \beta(vi)(d(u) + g(u)) + \alpha(viu) = 0 \quad (3.18)$$

for all $u, v, i \in L$. By applying hypothesis, we get $F(vi) + G(vi) = -\alpha(iv)$ and on using this in (3.18),

$$[\alpha(v), \alpha(i)]\alpha(u) + \beta(vi)(d(u) + g(u)) = 0. \quad (3.19)$$

Putting $2ji$ instead of i , we obtain that

$$\alpha(j)[\alpha(v), \alpha(i)]\alpha(u) + [\alpha(v), \alpha(j)]\alpha(i)\alpha(u) + \beta(v)\beta(ji)(d(u) + g(u)) = 0 \quad (3.20)$$

for all $u, v, i, j \in L$. By using (3.19), we get $\beta(ji)(d(u) + g(u)) = -[\alpha(j), \alpha(i)]\alpha(u)$ and using this in (3.20), $(\alpha(j)[\alpha(v), \alpha(i)] + [\alpha(v), \alpha(j)]\alpha(i) - \beta(v)[\alpha(j), \alpha(i)])\alpha(u) = 0$, for all $u, v, i, j \in L$. By taking $[u, r]$ instead of u , we find $(\alpha(j)[\alpha(v), \alpha(i)] + [\alpha(v), \alpha(j)]\alpha(i) - \beta(v)[\alpha(j), \alpha(i)])\alpha(r)\alpha(u) = 0$, for all $u, v, i, j \in L$. Thus

$$\alpha^{-1}(\alpha(j)[\alpha(v), \alpha(i)] + [\alpha(j), \alpha(i)]\alpha(v) - \beta(v)[\alpha(j), \alpha(i)])Ru = (0)$$

for all $u, v, i, j \in L$. Since $L = L^\star$, so

$$\alpha^{-1}(\alpha(j)[\alpha(v), \alpha(i)] + [\alpha(j), \alpha(i)]\alpha(v) - \beta(v)[\alpha(j), \alpha(i)])Ru^\star = (0)$$

for all $u, v, i, j \in L$. The $\star$ -primeness of R gives that

$$\alpha(j)[\alpha(v), \alpha(i)] + [\alpha(j), \alpha(i)]\alpha(v) - \beta(v)[\alpha(j), \alpha(i)] = 0.$$

Taking $2iv$ in place of i , we are left with

$$\alpha(i)[\alpha(j), \alpha(v)]\alpha(v) - \beta(v)\alpha(i)[\alpha(j), \alpha(v)] = 0. \quad (3.21)$$

Further, replacing i by $2ki$, we have

$$\alpha(ki)[\alpha(j), \alpha(v)]\alpha(v) - \beta(v)\alpha(ki)[\alpha(j), \alpha(v)] = 0$$

for all $v, i, j, k \in L$. From (3.21), $\alpha(i)[\alpha(j), \alpha(v)]\alpha(v) = \beta(v)\alpha(i)[\alpha(j), \alpha(v)]$ and combining this with the above equation, we get $[\alpha(k), \beta(v)]\alpha(i)[\alpha(j), \alpha(v)] = 0$. This implies that

$$\alpha^{-1}([\alpha(k), \beta(v)])L[j, v] = 0$$

for all $v, j, k \in L$. Since $L = L^\star$, therefore $\alpha^{-1}([\alpha(k), \beta(v)])L([j, v])^\star = (0)$, for all $j, k \in L$ and $v \in L \cap S_{a^\star}$. By Lemma 2.4, for each $v \in L \cap S_{a^\star}$, either $[\alpha(L), \beta(v)] = (0)$ or $[L, v] = (0)$. As $v + v^\star, v - v^\star \in L \cap S_{a^\star}$, for all $v \in L$. Therefore, either $[\alpha(L), \beta(v \pm v^\star)] = (0)$ or $[L, v \pm v^\star] = (0)$. This implies that for each $v \in L$, either $[\alpha(L), \beta(v)] = (0)$ or $[L, v] = (0)$ and by applying Brauer's

trick, either $[\alpha(L), \beta(L)] = (0)$ or $[L, L] = (0)$. By Lemma 2.9, the former case gives that $L \subseteq Z(R)$ and also by Lemma 2.5, $[L, L] = (0)$ implies $L \subseteq Z(R)$. In both cases, we get $L \subseteq Z(R)$, which contradiction to our assumption. Hence $L \subseteq Z(R)$ $\square$

Corollary 3.13. *Let F, G be two multiplicative generalized (α, β) -reverse derivations of R associated with (α, β) -reverse derivations d, g respectively such that $F(uv) + G(uv) + \alpha(vu) = 0$, for all $u, v \in R$. Then R is commutative.*

Corollary 3.14. *Let F, G be two multiplicative generalized reverse derivations of R associated with reverse derivations d, g respectively such that $F(uv) + G(uv) + vu = 0$, for all $u, v \in L$. Then $L \subseteq Z(R)$.*

Theorem 3.15. *If F is a multiplicative generalized (α, β) -reverse derivation of R associated with a nonzero (α, β) -reverse derivation d and α, β commute with $\star$ such that $[F(u) \circ \alpha(v), R] = (0)$, for all $u, v \in L$, then $L \subseteq Z(R)$*

Proof. Suppose that F is a multiplicative generalized (α, β) -reverse derivation of R associated with a nonzero (α, β) -reverse derivation d and α, β commute with $\star$ such that

$$[F(u) \circ \alpha(v), r] = 0 \quad (3.22)$$

for all $u, v \in L$ and $r \in R$. Suppose on the contrary that $L \not\subseteq Z(R)$. Replacing v by $2vi$ in (3.22), we have $0 = 2[F(u) \circ \alpha(vi), r] = 2[(F(u) \circ \alpha(v))\alpha(i) - \alpha(v)[F(u), \alpha(i)], r]$, for all $u, v, i \in L$ and $r \in R$. By using the fact that $\text{char}(R) \neq 2$, we get

$$[F(u) \circ \alpha(v), r]\alpha(i) + F(u) \circ \alpha(v)[\alpha(i), r] - [\alpha(v)[F(u), \alpha(i)], r] = 0.$$

By using (3.22), we find that $F(u) \circ \alpha(v)[\alpha(i), r] - [\alpha(v)[F(u), \alpha(i)], r] = 0$, for all $u, v, i \in L$ and $r \in R$. Taking $r\alpha(i)$ in place of r in last relation, we obtain $r[\alpha(v)[F(u), \alpha(i)], \alpha(i)] = 0$, for all $u, v, i \in L$ and $r \in R$. This implies that $zr[\alpha(v)[F(u), \alpha(i)], \alpha(i)] = 0$ and so

$$zR[\alpha(v)[F(u), \alpha(i)], \alpha(i)] = (0)$$

for all $u, v, i, z \in L$. As L is a $\star$ -Lie ideal of R , so $(z)^\star R[\alpha(v)[F(u), \alpha(i)], \alpha(i)] = (0)$, for all $u, v, i, z \in L$. The $\star$ -primeness of R gives that

$$0 = [\alpha(v)[F(u), \alpha(i)], \alpha(i)] = \alpha(v)[[F(u), \alpha(i)], \alpha(i)] + [\alpha(v), \alpha(i)][F(u), \alpha(i)] \quad (3.23)$$

for all $u, v, i \in L$, since L is a nonzero Lie ideal of R . By putting $2vk$ instead of v in the above equation, we have

$$\alpha(v)(\alpha(k)[[F(u), \alpha(i)], \alpha(i)] + [\alpha(k), \alpha(i)][F(u), \alpha(i)] + [\alpha(v), \alpha(i)]\alpha(k)[F(u), \alpha(i)] = 0$$

and by using (3.23), $[\alpha(v), \alpha(i)]\alpha(k)[F(u), \alpha(i)] = 0$, for all $u, v, i, k \in L$. This gives that

$$[v, i]L\alpha^{-1}([F(u), \alpha(i)]) = (0)$$

for all $u, v, i \in L$. Since L is a nonzero $\star$ -Lie ideal of R , therefore

$$([v, i])^\star L\alpha^{-1}([F(u), \alpha(i)]) = (0)$$

for all $u, v \in L$ and $i \in L \cap S_{a\star}(R)$. By Lemma 2.4 the previous equation gives that for each $i \in L \cap S_{a\star}(R)$ either $[L, i] = (0)$ or $[F(L), \alpha(i)] = (0)$. Let $i \in L$, as $i + (i)^\star, i - (i)^\star \in L \cap S_{a\star}(R)$ and then, either $[L, i \pm (i)^\star] = (0)$ or $[F(L), \alpha(i \pm (i)^\star)] = (0)$. Therefore, for each $i \in L$, either $[L, i] = (0)$ or $[F(L), \alpha(i)] = (0)$. This implies that either $[L, L] = (0)$ or $[F(L), \alpha(L)] = (0)$. The former case infers that $[L, L] = (0)$ and by Lemma 2.5, $L \subseteq Z(R)$, a contradiction to our supposition. From latter case we have

$$[F(u), \alpha(v)] = 0 \quad (3.24)$$

for all $u, v \in L$. By taking $2vu$ in place of u , we obtain that

$$0 = [\beta(u)d(v), \alpha(v)] = \beta(u)[d(v), \alpha(v)] + [\beta(u), \alpha(v)]d(v)$$

for all $u, v \in L$. Replacing u by $2iu$ yields $[\beta(i), \alpha(v)]\beta(u)d(v) = 0$, for all $u, v, i \in L$. This implies that

$$\beta^{-1}([\beta(i), \alpha(v)])L\beta^{-1}(d(v)) = (0)$$

for all $v, i \in L$. Since L is a $\star$ -Lie ideal R and α, β commute with $\star$, so

$$(\beta^{-1}([\beta(i), \alpha(v)]))^\star L\beta^{-1}(d(v)) = (0)$$

for all $i \in L$ and $v \in L \cap S_{a\star}(R)$. By Lemma 2.4 the last relation gives that, for each $v \in L \cap S_{a\star}(R)$, either $[\beta(L), \alpha(v)] = (0)$ or $d(v) = 0$. Application of similar techniques as used in the paragraph above to the equation (3.24) infers that either $[\beta(L), \alpha(L)] = (0)$ or $d(L) = (0)$.

By Lemma 2.9, the former case yields that $L \subseteq Z(R)$. Since d is nonzero, therefore by Lemma 2.6 the latter case also gives that $L \subseteq Z(R)$. In each case, we get $L \subseteq Z(R)$, a contradiction. Hence, $L \subseteq Z(R)$. $\square$

Acknowledgement. The second author gratefully acknowledges the financial assistance by UGC.

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¹ DEPARTMENT OF MATHEMATICS AND STATISTICS, HIMACHAL PRADESH UNIVERSITY, SUMMER HILL, SHIMLA, 171005, INDIA
Email address: mpatial.math@gmail.com

²DEPARTMENT OF MATHEMATICS AND STATISTICS, HIMACHAL PRADESH UNIVERSITY, SUMMER HILL, SHIMLA, 171005, INDIA
Email address: geetasharmamath@gmail.com